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# MATHEMATICS FOR TECHNICAL STUDENTS

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## PART II

*WITH DIAGRAMS*

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# PREFACE

This series is designed to provide the basic mathematical equipment for technical students. The range of work is that covered by national certificate courses in engineering, building and chemistry. Each of the three volumes will include one year's work, but the second and third will contain sufficient revision to allow for variations in the syllabuses of different technical institutions. It is hoped that the series may also prove useful in secondary schools with a technical side.

Short historical notes are included as they may help to develop in the student a further interest in the subject.

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H. A. H.

# AUTHORS' FOREWORD TO PART II

This volume is designed to cover the second year of a three years' national certificate course. It begins with a revision chapter in algebra, which, however, contains some harder examples than in Part I, and the extension of the use of symbols to include units.

Teachers may not wish to include the proofs of theorems given in Chapters VI and VII, but the results of these chapters are required for use later. Sufficient solid geometry has been introduced to enable the student to solve trigonometrical problems in three dimensions and harder problems in mensuration.

Although no previous knowledge of trigonometry is assumed in this volume, an easier introduction is provided by Chapters XVI and XVII of Part I.

The chapters on differentiation and integration are mainly graphical, but they include easy applications of the rules for differentiating and integrating a power of $x$.

The order of arrangement of the chapters is that which is most convenient for reference, but it is not intended that they should necessarily be read in that order. A suggested order of reading is as follows: Chapters I, IX (pp. 193–213), II, III, IV, VI, VII, VIII, IX (pp. 214–224), X, XI, V, XV, XVI, XII, XIII, XIV.

The authors have aimed at including all the work likely to be done in any technical institution in the second year of a course of this type, though it is not necessarily expected that any single institution would cover the whole of it in one year. If the whole volume cannot be covered the following chapters may be omitted: Chapters VIII, IX (pp. 214–224), XIII (pp. 292–303), XIV (pp. 320–340), XVI (pp. 384–398).

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# CONTENTS

# ALGEBRA

## CHAPTER I

## ALGEBRAIC PROCESSES, MAINLY REVISION

### Powers of a number

The meanings of integral and fractional powers of a number, and the rules for multiplying and dividing powers of a number are summarized below.

If $m$ is a positive integer (whole number),

$a^m$ means $a \times a \times a \times a$ . . . . . . . . . . $m$ factors.

$$\begin{aligned} \therefore\ a^3 \times a^4 &= (a \times a \times a) \times (a \times a \times a \times a) \\ &= a \times a \times a \times a \times a \times a \times a \quad \text{. . . . . . 7 factors.} \\ &= a^7 = a^{3+4}. \end{aligned}$$

In the same way, if $m$ and $n$ are any positive integers,

$$\underline{a^m \times a^n = a^{m+n}}. \quad \text{(Rule I).}$$

Also
$$\begin{aligned} \frac{a^6}{a^2} &= \frac{a \times a \times a \times a \times a \times a}{a \times a} \\ &= a \times a \times a \times a \quad \text{. . . . . . 4 factors.} \end{aligned}$$

$$\therefore\ \frac{a^6}{a^2} = a^4 = a^{6-2}.$$

In the same way, if $m$ is a positive integer greater than $n$,

$$\underline{\frac{a^m}{a^n} = a^{m-n}}. \quad \text{(Rule II).}$$

Again
$$\begin{aligned} (a^2)^3 &= (a \times a) \times (a \times a) \times (a \times a) \\ &= a \times a \times a \times a \times a \times a \quad \text{. . . 6 factors.} \end{aligned}$$

$$\therefore\ (a^2)^3 = a^6 = a^{2 \times 3}.$$

In the same way, if $m$ and $r$ are positive integers,

$$\underline{(a^m)^r = a^{mr}.} \quad \text{(Rule III).}$$

## Fractional powers

If we assume that Rule III applies when $m$ is a fraction,

$$(a^{\frac{2}{3}})^3 = a^{\frac{2}{3}\times 3} = a^2$$

$$\therefore\ a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

Also $$a^{\frac{2}{3}} = a^{2\times\frac{1}{3}} = (a^{\frac{1}{3}})^2 = (\sqrt[3]{a})^2.$$

In the same way

$$\underline{a^{\frac{p}{q}} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p,}$$

where $\sqrt[q]{a^p}$ means the number whose $q$th power is $a^p$, i.e. the $q$th root of $a^p$.

Fractional powers defined in this way obey Rules I and II. For instance if

$$x = a^{\frac{2}{3}} \times a^{\frac{1}{2}},$$

$$x^6 = (a^{\frac{2}{3}})^6 \times (a^{\frac{1}{2}})^6 = a^{\frac{2}{3}\times 6} \times a^{\frac{1}{2}\times 6} = a^4 \times a^3 = a^7$$

$$\therefore\ x = \sqrt[6]{a^7} = a^{\frac{7}{6}} = a^{\frac{2}{3}+\frac{1}{2}}$$

$$\therefore\ a^{\frac{2}{3}} \times a^{\frac{1}{2}} = a^{\frac{2}{3}+\frac{1}{2}}.$$

Hence the product is formed by using Rule I.

## Negative powers and $a^0$

If we assume Rule I applies to negative indices as well as positive indices

$$a^3 \times a^{-2} = a^{3+(-2)} = a^{3-2} = a$$

$$\therefore\ a^{-2} = \frac{a}{a^3} = \frac{1}{a^2}.$$

Again $$a^3 \times a^{-\frac{9}{4}} = a^{3-\frac{9}{4}} = a^{\frac{3}{4}}$$

$$\therefore\ a^{-\frac{9}{4}} = \frac{a^{\frac{3}{4}}}{a^3} = \frac{1}{a^{\frac{9}{4}}}.$$

In the same way, if $n$ is any number,

$$a^{-n}=\frac{1}{a^n}.$$

Also by Rule I, $a^0=a^{2-2}=a^2\times a^{-2}=\frac{a^2}{a^2}$

$$\therefore \; a^0=1.$$

**Powers of *ab***

$(4\times 9)^{\frac{1}{2}}=36^{\frac{1}{2}}=6=2\times 3=4^{\frac{1}{2}}\times 9^{\frac{1}{2}}.$

In the same way $(ab)^n=a^n\times b^n=a^nb^n$. This should not be confused with $ab^n$, which means $a\times b^n$.

**Roots and Surds**

$\sqrt[n]{a}$ stands for the positive number whose $n$th power is $a$. Since $(-1)(-1)=1$, it follows that when $n$ is even the $n$th power of $(-\sqrt[n]{a})$ is also $a$.

For instance, if $x^4=16$

$$x=\sqrt[4]{16} \text{ or } -\sqrt[4]{16} \qquad \text{i.e. } x=2 \text{ or } -2.$$

A root can often be expressed in a simpler form by factorizing the number under the root sign.

*Example.*—Simplify (i) $\sqrt{128}$; (ii) $\sqrt[3]{189}$.

(i) $\sqrt{128}=\sqrt{64\times 2}=\sqrt{64}\,\sqrt{2}=8\sqrt{2}.$

(ii) $\sqrt[3]{189}=\sqrt[3]{27\times 7}=\sqrt[3]{27}\,\sqrt[3]{7}=3\sqrt[3]{7}.$

If $a$ is not a perfect square $\sqrt{a}$ is called a surd. Thus $\sqrt{2}$ is a surd, but $\sqrt{16}$ is not. If $a$ is not the $n$th power of a whole number or of a fraction $\sqrt[n]{a}$ is a surd.

The quickest way to calculate the reciprocal of a surd $\sqrt{a}$ is to multiply numerator and denominator by $\sqrt{a}$.

*Example.*—Express $\frac{1}{2\sqrt{3}}$ without a surd in the denominator and as a decimal.

$$\frac{1}{2\sqrt{3}}=\frac{\sqrt{3}}{2(\sqrt{3})^2}=\frac{\sqrt{3}}{2\times 3}=\frac{\sqrt{3}}{6}\simeq\frac{1\cdot 732}{6}\simeq 0\cdot 289.$$

*Examples.*—Simplify (a) $\dfrac{a^4(b^2c^3)^3}{(a^2bc)^6}$; (b) $\dfrac{(\sqrt{2fp})^3}{4f^2p}$; (c) $\dfrac{k^{1\cdot9}}{k^{-0\cdot7}}$.

$$(a)\quad \frac{a^4(b^2c^3)^3}{(a^2bc)^6}=\frac{a^4b^{2\times3}c^{3\times3}}{a^{2\times6}b^6c^6}=\frac{a^4b^6c^9}{a^{12}b^6c^6}=\frac{c^{9-6}}{a^{12-4}}=\frac{c^3}{a^8}.$$

$$(b)\quad \frac{(\sqrt{2fp})^3}{4f^2p}=\frac{(2^{\frac12}f^{\frac12}p^{\frac12})^3}{4f^2p}=\frac{2^{\frac32}f^{\frac32}p^{\frac32}}{4f^2p}=\frac{2^{\frac32}p^{\frac32-1}}{2^2 . f^{2-\frac32}}$$

$$=\frac{p^{\frac12}}{2^{\frac12}f^{\frac12}} \text{ or } \sqrt{\frac{p}{2f}}.$$

$$(c)\quad \frac{k^{1\cdot9}}{k^{-0\cdot7}}=k^{1\cdot9}\times k^{0\cdot7}=k^{1\cdot9+0\cdot7}=k^{2\cdot6}.$$

*Example.*—From the formula for the volume of a sphere, $V=\frac43\pi r^3$, find $r$ in terms of $\pi$ and V.

$$4\pi r^3=3V$$

$$\therefore\ r^3=\frac{3V}{4\pi}$$

$$\therefore\ r=\sqrt[3]{\frac{3V}{4\pi}} \text{ or } \left(\frac{3V}{4\pi}\right)^{\frac13}.$$

## Exercise I

1. Simplify each of the following and find its value to three significant figures, given that $\sqrt2=1\cdot414$, $\sqrt3=1\cdot732$, $\sqrt5=2\cdot236$, $\sqrt[3]{2}=1\cdot260$.

(a) $\sqrt{27}$ (b) $\sqrt[3]{54}$ (c) $\sqrt{80}$ (d) $\sqrt{147}$ (e) $\sqrt{288}$.

2. Express without a surd in the denominator and also as a decimal:

(a) $\dfrac{1}{\sqrt2}$ (b) $\dfrac{1}{2\sqrt5}$ (c) $\dfrac{6}{\sqrt3}$ (d) $\dfrac{10}{\sqrt5}$.

3. Express with positive indices instead of root signs:

(a) $\sqrt7$ (b) $\sqrt[3]{3}$ (c) $\dfrac{1}{\sqrt[4]{2}}$ (d) $\sqrt{\dfrac23}$ (e) $\sqrt{5^7}$.

4. Express with positive indices instead of root signs:

(a) $\sqrt{t}$ (b) $\sqrt{x^3}$ (c) $(\sqrt{y})^5$ (d) $\sqrt{4m}$ (e) $\sqrt[3]{x^2y^2}$.

5. Simplify:

(a) $\left(\frac{2}{a}\right)^2 \div \left(\frac{1}{2a}\right)^3$ (b) $\frac{(3x)^4}{3x^4}$ (c) $\frac{(xy^2)^3}{(xy^3)^2} \times \left(\frac{y}{x}\right)^4$.

6. Simplify:

(a) $\frac{\left(\frac{m}{2n}\right)^2}{2m^2n^3}$ (b) $\frac{\frac{3}{15}\pi r^5}{\frac{4}{3}\pi r^3}$ (c) $\frac{\frac{1}{2}\frac{w}{g}(\sqrt{2gh})^2}{Wh}$.

7. Which of $\frac{2p^2}{2p^3}$ and $\frac{(2p)^2}{(2p)^3}$ equals $\frac{1}{2p}$?

8. Which of $\frac{a^2x^2}{ax^3}$ and $\frac{ax^2}{(ax)^3}$ equals $\frac{a}{x}$?

9. Express as a power of $x$:

(a) $\sqrt{x^7}$ (b) $\sqrt[3]{\frac{1}{x^2}}$ (c) $\frac{x^3}{\sqrt{x^5}}$ (d) $\frac{x^{2\cdot7}}{\sqrt{x^{1\cdot8}}}$.

Simplify:

10. (a) $16^{\frac{1}{4}}$ (b) $y^{\frac{1}{5}} \times y^{\frac{3}{10}}$ (c) $\sqrt[3]{27a^2b}$ (d) $16a^{\frac{1}{2}} \times \left(\frac{a}{2}\right)^4$.

11. (a) $\left(\frac{27}{8}\right)^{-\frac{4}{3}}$ (b) $\frac{h^4}{8r^3} \times \left(\frac{2r}{h}\right)^4$ (c) $(81x)^{\frac{3}{4}} \times (8x)^{\frac{4}{3}}$

(d) $\frac{p^{1\cdot7} \times q^{-0\cdot6}}{p^{-0\cdot3} \times q^{0\cdot4}}$.

12. $\frac{p^{\frac{1}{3}} \times p^{\frac{1}{4}}}{p^{-\frac{1}{12}}}$ (b) $10a^{-\frac{5}{2}}(25a^5)^{\frac{1}{2}}$ (c) $\sqrt{\frac{a^5b^{-3}}{a^{\frac{3}{2}}b^{\frac{3}{4}}}}$.

13. (a) $(v^{2n})^{-\frac{1}{n}}$ (b) $\sqrt[3]{m^{\frac{3}{2}}n^{-\frac{9}{2}}}$ (c) $\frac{\sqrt{qp^{2n}r^n}}{3\left(\frac{p}{r}\right)^n}$.

Express without fractional or negative indices:

14. (a) $\sqrt{y^{-1}}$ (b) $\left(\frac{p}{q}\right)^{-2}$ (c) $\left(\frac{x}{y}\right)^{-\frac{1}{3}}$ (d) $ml^{-1}t^{-2}$.

15. (a) $\frac{(lt^{-1})^2}{lt^{-2}}$ (b) $x^{\frac{1}{4}}y^{-\frac{1}{2}}$ (c) $\frac{\sqrt{3p^{2k}q^{k-\frac{1}{2}}}}{(pq^{\frac{1}{2}})^{k+\frac{1}{2}}}$.

**16.** If $p_1v_1^n = p_2v_2^n$ prove $p_1v_1\left\{1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{n}}\right\} = p_1v_1 - p_2v_2$.

**17.** If $y^4 = 16x$, express $y$ in terms of $x$.

**18.** From the formula for the volume of a cylinder $V = \pi r^2 h$ express $r$ in terms of V and $h$.

**19.** If $\frac{1}{m^3} = \frac{a^2}{b}$, prove $m = a^{-\frac{2}{3}}b^{\frac{1}{3}}$.

**20.** If $\pi Eyd^4 = 8wl^4$, prove $d = l\left(\frac{8w}{\pi Ey}\right)^{\frac{1}{4}}$.

## Brackets and products

By putting an expression between two brackets we imply that the expression is to be treated as a whole; thus $3(2x+4)$ means that each term of $(2x+4)$ is to be multiplied by 3.

$\therefore\ 3(2x+4) = 3 \times 2x + 3 \times 4 = 6x + 12.$

*Example.*—Remove the brackets from (*a*) $\frac{1}{3}h^2(3r-h)$; (*b*) $t^{\frac{1}{4}}(2\sqrt{t}-3t)$.

(*a*) $\frac{1}{3}h^2(3r-h) = \frac{1}{3}h^2 \times 3r - \frac{1}{3}h^2 \times h = h^2 - \frac{1}{3}h^3.$

(*b*) $t^{\frac{1}{4}}(2\sqrt{t}-3t) = t^{\frac{1}{4}} \times 2t^{\frac{1}{2}} - t^{\frac{1}{4}} \times 3t = 2t^{\frac{3}{4}} - 3t^{\frac{5}{4}}.$

*Example.*—Expand the products (*a*) $(p-2q)(p^2+pq)$; (*b*) $(2y-3)(y^2+4y-6)$.

(*a*)
$$\begin{aligned}(p-2q)(p^2+pq) &= p(p^2+pq) - 2q(p^2+pq)\\ &= p^3 + p^2q - 2p^2q - 2pq^2\\ &= p^3 - p^2q - 2pq^2.\end{aligned}$$

(*b*)
$$\begin{aligned}(2y-3)(y^2+4y-6) &= 2y(y^2+4y-6) - 3(y^2+4y-6)\\ &= 2y^3 + 8y^2 - 12y - 3y^2 - 12y + 18\\ &= 2y^3 + 5y^2 - 24y + 18.\end{aligned}$$

Any multiplication can be set out as in arithmetic. For instance, example (*b*) above can be set out as follows:

$$\begin{array}{rl} y^2 + 4y - 6 & \\ 2y - 3 & \\ \hline 2y^3 + 8y^2 - 12y \qquad & \{\text{this line is } 2y(y^2+4y-6)\} \\ -3y^2 - 12y + 18 & \{\text{this line is } -3(y^2+4y-6)\} \\ \hline 2y^3 + 5y^2 - 24y + 18 & \end{array}$$

*Example.*—Form the product $(t^{1\cdot6}+3)(2t^{1\cdot2}-t^{-0\cdot4})$.

$$\text{Expression} = t^{1\cdot6}(2t^{1\cdot2}-t^{-0\cdot4})+3(2t^{1\cdot2}-t^{-0\cdot4})$$
$$= t^{1\cdot6}\times 2t^{1\cdot2}-t^{1\cdot6}t^{-0\cdot4}+3\times 2t^{1\cdot2}-3t^{-0\cdot4}$$
$$= 2t^{2\cdot8}-t^{1\cdot2}+6t^{1\cdot2}-3t^{-0\cdot4}$$
$$= 2t^{2\cdot8}+5t^{1\cdot2}-3t^{-0\cdot4}.$$

The effect of a $-$ sign in front of a bracket is to change all the signs of the terms within the bracket when it is removed.

*Example.*—Simplify $2p(r-s)-s(r-2p)-\frac{1}{2}r(4p-s)$.

$$\text{Expression} = 2pr-2ps-sr+2sp-2rp+\tfrac{1}{2}rs$$
$$= 2pr-2ps-rs+2ps-2pr+\tfrac{1}{2}rs.$$
$$= -\tfrac{1}{2}rs.$$

**Some important products**

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ab+b^2$$
$$\therefore\ \underline{(a+b)^2=a^2+2ab+b^2} \quad . \quad . \quad . \quad . \quad \text{I.}$$

In the same way

$$\underline{(a-b)^2=a^2-2ab+b^2} \quad . \quad . \quad . \quad . \quad \text{II.}$$
$$\underline{(a+b)(a-b)=a^2-b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{III.}$$

These results should be memorized. By using them the square of any expression containing two terms and the product of the sum and difference of two terms can be written down.

*Examples.*—Express without brackets (*a*) $(2x-3y)^2$; (*b*) $(5r-4s)(5r+4s)$; (*c*) $(2m^{0\cdot8}+m^{0\cdot2})^2$; (*d*) $(f+g-h)(f+g+h)$.

(*a*) $(2x-3y)^2=(2x)^2-2.2x.3y+(3y)^2$ [by II above]
$=4x^2-12xy+9y^2.$

(*b*) $(5r-4s)(5r+4s)=(5r)^2-(4s)^2$ [by III above]
$=25r^2-16s^2.$

(*c*) $(2m^{0\cdot8}+m^{0\cdot2})^2=(2m^{0\cdot8})^2+2.2m^{0\cdot8}m^{0\cdot2}+(m^{0\cdot2})^2$ [by I above]
$=4m^{1\cdot6}+4m+m^{0\cdot4}$

(*d*) $(f+g-h)(f+g+h)=(f+g)^2-h^2$ [by III with $a=f+g$, $b=h$]
$=f^2+2fg+g^2-h^2.$

An application of the formula for $(a+b)(a-b)$ occurs in calculating the values of expressions like $\frac{1}{5-3\sqrt{2}}$.

$(5-3\sqrt{2})(5+3\sqrt{2})=25-(3\sqrt{2})^2=25-18=7$, and hence does not contain the surd $\sqrt{2}$. Therefore, multiplying numerator and denominator of $\frac{1}{5-3\sqrt{2}}$ by $(5+3\sqrt{2})$, we get

$$\frac{1}{5-3\sqrt{2}}=\frac{5+3\sqrt{2}}{(5-3\sqrt{2})(5+3\sqrt{2})}=\frac{5+3\sqrt{2}}{7}\simeq\frac{9{\cdot}242}{7}=1{\cdot}320.$$

### Simplification of expressions containing products

$(p-2q)(p-3q)$ means that the expressions $(p-2q)$ and $(p-3q)$ are to be multiplied and, since multiplication takes precedence over addition or subtraction, the product must be found first and the resulting terms added or subtracted.

*Example.*—Simplify $(3p+q)^2-(p-2q)(p-3q)-(p-q)^2$.

Expression

$$\begin{aligned}&=9p^2+6pq+q^2-(p^2-5pq+6q^2)-(p^2-2pq+q^2)\\&=9p^2+6pq+q^2-p^2+5pq-6q^2-p^2+2pq-q^2\\&=(9-1-1)p^2+(6+5+2)pq+(1-6-1)q^2\\&=7p^2+13pq-6q^2.\end{aligned}$$

### Addition and subtraction of fractions

Find the L.C.M. of the denominators of the fractions and express each fraction with this L.C.M. as denominator. Then all the fractions have a common denominator and the numerators can be added or subtracted. Remember that the bar of a fraction is equivalent to a bracket.

*Example.*—Simplify $\frac{6r+5}{6s}-\frac{2rs+r}{2s^2}$.

$$6s=2\times3\times s$$
$$2s^2=2\times s\times s.$$

$\therefore$ L.C.M. $= 2 \times 3 \times s^2 = 6s^2$.

$$\therefore \text{ expression} = \frac{s(6r+5)}{6s^2} - \frac{3(2rs+r)}{6s^2} = \frac{s(6r+5)-3(2rs+r)}{6s^2}$$

$$= \frac{6rs+5s-6rs-3r}{6s^2} = \frac{5s-3r}{6s^2}.$$

## Exercise II

1. Remove brackets from :

   (a) $2(x^2-4x)$ (b) $M(2M+3N)$ (c) $2\left(\omega t+\frac{2\pi}{3}\right)$

   (d) $a^2(a-h)$.

2. Add :

   (a) $x^2+2x-1$, $3x-4$, $2x^2+5$.
   (b) $\theta+\alpha$, $2\alpha-\theta$, $3\theta+4\alpha$.
   (c) $y^2-a^2$, $4y^2+ay$, $y^2+ay+3a^2$.
   (d) $r_1l_1+r_2l_2$, $r_1l_1-r_3l_3$, $-r_2l_2+r_3l_3$.

3. Add :

   (a) $x^2+ax+bx+ab$, $a^2-ab-ax$, $b^2-bx$.
   (b) $p^2+2pq+q^2$, $p^2-2pq+q^2$, $2(p^2-q^2)$.
   (c) $3(m^2-2mn-3n^2)$, $4(n^2+mn-m^2)$.

4. Subtract :

   (a) $m+2n$ from $3m+4n$.
   (b) $4x-3y$ from $x+2y$.
   (c) $a^2-2ab+b^2$ from $a^2+2ab+b^2$.
   (d) $nt+30$ from $2nt-60$.
   (e) $ab+2b^2+ca$ from $a^2+ab+b^2$.

5. Remove the brackets and simplify :

   (a) $m-n+2(3m+2n)$.
   (b) $4(p-q)-2(p+q)$.
   (c) $2(x^2+2ax+a^2)-(x^2-a^2)$.
   (d) $2(t^3+1{\cdot}4t^2-2{\cdot}7t)-4(0{\cdot}5t^3-t^2+1{\cdot}3t)$.

6. Remove the brackets and simplify :

   (a) $3a^2(a-h)-a(a^2-h^2)-2ah(a+h)$.
   (b) $2(1-\cos\theta)-3(2-3\cos\theta)$.
   (c) $2(1+\sin A)-3(2\sin A+\cos A)+4(\sin A-\cos A)$.
   (d) $\sin x(1-\cos x)-\cos x(1-\sin x)$.

7. Remove the brackets and simplify :

(a) $t^7(1-5t)+t^6(t-t^2+1)$.
(b) $2a+b-c-\{a-2(b-c)\}+\{3a-(b+4c)\}$.
(c) $2[1-x-\{3-2(1-x)\}]$.
(d) $3\{5\cos x-2(\sin x+\cos x)\}-6(\cos x-\sin x)$.

8. Multiply :

(a) $2x-3$ by $3x+4$ (b) $l^2+m^2$ by $l-2m$
(c) $4-x$ by $2-3x$ (d) $p^2-pq$ by $2p+q$.

9. Multiply :

(a) $2x^2-3x+4$ by $2x-4$ (b) $x^2-x-7$ by $x^2+2x+3$
(c) $p+lm$ by $l+mp$ (d) $1-e+e^2$ by $1+e+e^2$.

10. Multiply by picking out like products :

(a) $(x+1)(x+4)$ (b) $(a+2b)(a-3b)$
(c) $(p+q)(p-4q)$ (d) $(2y-7)(y+5)$
(e) $(2x^2+x-1)(3x+2)$ (f) $(l^2-lm)(l+2m)$.

11. Expand the following squares :

(a) $(1+x)^2$ (b) $(2y+3)^2$ (c) $(p-2q)^2$ (d) $(2ar-5s)^2$.

12. Express without brackets :

(a) $(m+n)(m-n)$ (b) $\left(\frac{r}{s}+1\right)\left(\frac{r}{s}-1\right)$
(c) $(a+b+c)(a-b-c)$.

13. Evaluate by expressing each number as a sum or difference as in (a) :

(a) $169^2=170^2-2.170+1$ (b) $134^2$ (c) $98^2$ (d) $47^2$.

14. Simplify

(a) $\frac{6}{a}-\frac{1}{2a}-\frac{13}{3a}$ (b) $\frac{2(x+y)}{7}-\frac{x+4y}{14}$
(c) $\frac{a+b}{ab}-\frac{a-b}{a^2}$ (d) $\frac{6xy+1}{y^2}-\frac{2(3x+1)}{y}$.

15. Express without a surd in the denominator and also as a decimal :

(a) $\frac{1}{\sqrt{2}+1}$ (b) $\frac{25}{4-\sqrt{6}}$ (c) $\frac{6}{5-\sqrt{7}}$.

16. Find the value of $20-\sqrt{399}$ given that $\sqrt{399}=19{\cdot}975$. Also by writing $20-\sqrt{399}=\frac{(20-\sqrt{399})(20+\sqrt{399})}{20+\sqrt{399}}$ find its value to four sig. fig. (The student will notice that the first method gives two sig. fig. only, whereas the second gives four sig. fig.)

17. By the method of Question 16 find $17 - \sqrt{288}$ to four sig. fig.

Simplify :

18. (a) $x(x+y)+y(x+y)$ (b) $(a+b)^3-(a-b)^2$.

19. (a) $(x+y)(x-y)-x(x+2y)$ (b) $(1+\sin\theta)(1-\sin\theta)$.

20. (a) $(2+\sin x)^2+(2-\sin x)^2$ (b) $\frac{1}{2}w(l-x)^2-\frac{1}{2}wl(l-x)$.

21. (a) $(2x+1)(x-1)-(3x+2)(x-2)$
(b) $x(x-1)(x-2)-(5-3x)(x+1)$.

22. By multiplying $(a+b)^2$ by $(a+b)$ prove that

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3,$$

and hence find $(2y+3)^3$ in powers of $y$.

23. Find a formula for $(a-b)^3$ by the method used in Question 22, and use it to expand $(k-1)^3$ in powers of $k$.

24. It can be shown that the volume V cu. ft. of a segment of a sphere of radius $r$ ft. and height $h$ ft. is given by

$$V=\tfrac{2}{3}\pi r^3-\pi r^2(r-h)+\tfrac{1}{3}\pi(r-h)^3$$

Prove that $\quad V=\frac{1}{3}\pi h^2(3r-h)$.

25. If $A=akk'+k+k'$, $B=ak'+1$, $C=ak+1$, $D=a$, prove that $BC-AD=1$.

## Equations

Generally, if an equation contains only one letter or symbol, that letter must have a definite numerical value or values. For instance, if $8x=5$, $x$ must be $\frac{5}{8}$, and if $(x-1)(x+2)=0$; $x$ must be 1 or $-2$. Neither of these equations is a true statement for any other value of $x$. The numerical values which the unknown letter must have, are called the "roots" of the equation and the process of finding them is called "solving the equation." The *degree of an equation is the highest power of the unknown letter that occurs in it.* The equations $2x+1=7$, $t^2-t-4=0$, $p^4=16$ are of the first, second and fourth degrees in $x$, $t$ and $p$ respectively.

An equation or formula may contain several letters, and then any particular letter can only have a certain value or values in terms of the other letters. For instance, if $v^2=2gh$, the

equations $g=\frac{v^2}{2h}$ and $h=\frac{v^2}{2g}$ express $g$ and $h$ in terms of the other two letters. The process of solving an equation to find a particular letter in terms of other letters is called making that letter the subject of the formula.

### Simple equations

An equation or formula in which the unknown letter occurs to the first degree is called a simple equation.

Corresponding examples of solving an equation and of changing the subject of a formula are shown below side by side.

*Example.*—

Solve the equation:

$$2(x-13)=10-3(x+2)$$
$$2x-26=10-3x-6$$
$$\therefore\ 2x+3x=26+10-6$$
$$\therefore\ 5x=30$$
$$\therefore\ x=6$$

Make $x$ the subject of the formula:

$$2(x-b)=5a-3(x+a)$$
$$2x-2b=5a-3x-3a$$
$$\therefore\ 2x+3x=5a-3a+2b$$
$$\therefore\ 5x=2a+2b$$
$$\therefore\ x=\frac{2a+2b}{5}.$$

*Example.*—

Solve the equation:

$$\frac{t+1}{3}-\frac{5t+1}{12}=\frac{t}{2}$$

Multiply by 12 the L.C.M. of the denominators:

$$4(t+1)-(5t+1)=6t$$
$$\therefore\ 4t+4-5t-1=6t$$
$$\therefore\ 4t-5t-6t=1-4$$
$$\therefore\ -7t=-3$$
$$\therefore\ t=\frac{-3}{-7}=\frac{3}{7}.$$

Make $t$ the subject of the formula:

$$\frac{t+p}{q}-\frac{5t+p}{4q}=\frac{t}{2}$$

Multiply by $4q$ the L.C.M. of the denominators:

$$4(t+p)-(5t+p)=2qt$$
$$\therefore\ 4t+4p-5t-p=2qt$$
$$\therefore\ 4t-5t-2qt=-4p+p$$
$$\therefore\ -(1+2q)t=-3p$$
$$\therefore\ t=\frac{-3p}{-(1+2q)}=\frac{3p}{1+2q}.$$

**Equations involving surds**

If one term of an equation contains a square root, write the equation with that term on one side and then square both sides.

*Example.*—Make $x$ the subject of the formula :

$$\sqrt{\frac{x+a}{2r}}-\frac{a}{r}=0.$$

Adding $\frac{a}{r}$ to both sides :

$$\sqrt{\frac{x+a}{2r}}=\frac{a}{r}.$$

Squaring both sides :

$$\frac{x+a}{2r}=\frac{a^2}{r^2}.$$

Multiplying throughout by $2r$ :

$$x+a=\frac{2ra^2}{r^2}=\frac{2a^2}{r}$$

$$\therefore\ x=\frac{2a^2}{r}-a.$$

**Identities**

Some equations are true for every value of the letters. For instance, if we remove the bracket in the equation

$$x+2(x-2)=3x-4,$$

we get

$$x+2x-4=3x-4$$

i.e. $$3x-4=3x-4.$$

Since $3x=3x$ and $4=4$ whatever value $x$ has, the equation is true for all values of $x$.

An equation which is true for every value of the unknown letter or letters is called an "identity." We sometimes write $\equiv$ for "is identically equal to" instead of $=$ in identities. For instance, $(a+b)^2\equiv a^2+2ab+b^2$.

*Example.*—Prove $(ab-cd)^2+(bc+ad)^2 \equiv (a^2+c^2)(b^2+d^2)$.

$$\begin{aligned}(ab-cd)^2+(bc+ad)^2 &\equiv a^2b^2-2abcd+c^2d^2+b^2c^2+2abcd+a^2d^2\\ &\equiv a^2b^2+a^2d^2+c^2b^2+c^2d^2\\ &\equiv a^2(b^2+d^2)+c^2(b^2+d^2)\\ &\equiv (a^2+c^2)(b^2+d^2).\end{aligned}$$

## Problems involving simple equations and formulæ

To solve a problem in which one unknown quantity occurs we denote that quantity by some letter, often the first letter of the quantity, for instance, $r$ for radius, A for area, etc. Then we form an equation in this letter from the given facts.

*Example.*—A hollow iron pipe 2 ft. long and 560 lb. wt. per cu. ft. weighs 2·65 lb. wt. It is $\frac{1}{10}$th in. thick. Find its internal radius.

If the internal radius is $r$ in., the outside radius is $(r+\frac{1}{10})$ in. and the area of a cross section of the pipe is

$$\pi\{(r+\tfrac{1}{10})^2-r^2\}\text{ sq. in.} = \frac{\pi}{144}\left(r^2+\frac{r}{5}+\frac{1}{100}-r^2\right)\text{ sq. ft.}$$

$$=\frac{\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right)\text{ sq. ft.}$$

Therefore the volume of iron used is :

$$\frac{2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right)\text{ cu. ft.}$$

And hence its weight is :

$$\frac{560\times 2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right)\text{ lb. wt.}$$

$$\therefore \frac{560\times 2\pi}{144}\left(\frac{r}{5}+\frac{1}{100}\right)=2{\cdot}65$$

$$\therefore \frac{70\pi}{9}\left(\frac{r}{5}+\frac{1}{100}\right)=2{\cdot}65$$

$$\therefore \frac{r}{5}+\frac{1}{100}=\frac{9\times 2\cdot 65}{70\pi}=0\cdot 1085$$

$$\therefore \frac{r}{5}\simeq 0\cdot 098\dot{5}$$

$$\therefore \text{radius}\simeq 0\cdot 49 \text{ in. to } \tfrac{1}{100}\text{th in.}$$

*Example.*—What weight of liquid of specific gravity $s_1$ must be mixed with $n$ kg. of a liquid of specific gravity $s_2$ to give a mixture of specific gravity $s$ ?

Let the weight be $x$ kg. Then the volumes of the two liquids are :

$$\frac{1000x}{s_1} \text{ and } \frac{1000n}{s_2} \text{ cu. cm.}$$

$$\therefore \text{Volume of mixture}=\left(\frac{1000x}{s_1}+\frac{1000n}{s_2}\right) \text{ cu. cm.}$$

$$\text{Weight of mixture}=1000(x+n) \text{ gm.}$$

$$\therefore s=\frac{1000\,(x+n)}{\left(\frac{1000x}{s_1}+\frac{1000n}{s_2}\right)}=\frac{x+n}{\frac{x}{s_1}+\frac{n}{s_2}}$$

Multiplying by $\frac{x}{s_1}+\frac{n}{s_2}$:

$$\frac{sx}{s_1}+\frac{sn}{s_2}=x+n$$

$$\therefore ss_2x+ss_1n=s_1s_2x+s_1s_2n$$

$$\therefore (ss_2-s_1s_2)x=(s_1s_2-ss_1)n$$

$$\therefore x=\frac{(s_1s_2-ss_1)n}{ss_2-s_1s_2}$$

$$\therefore \text{weight of liquid}=\frac{(s_1s_2-ss_1)n}{ss_2-s_1s_2} \text{ kg.}$$

## Exercise III

**1.** Solve the equations :

(a) $2(x+1)=5x-7$ (b) $\dfrac{m-2}{5}=m-4$

(c) $\dfrac{y}{3}=\dfrac{2y-1}{6}+\dfrac{y-1}{4}$ (d) $\dfrac{1}{2t}=\dfrac{1}{3}$.

(e) $(t+2)^2=(t-2)^2+6$ (f) $\dfrac{x-1}{x+2}=\dfrac{1}{4}$.

**2.** Express the letter in brackets in terms of the other letters :

(a) $3p-q-r=2(p+q+r)$ [$p$] (b) $\frac{1}{2}(p+q)=p-q$ [$q$]

(c) $\dfrac{R-2r}{R}=\dfrac{3}{5}$ [R] (d) $S=2\pi rh+\pi r^2$ [$h$]

(e) $y=\dfrac{4Wl^3}{Ebd^3}$ [E] (f) $\dfrac{1}{u}+\dfrac{1}{v}=\dfrac{1}{f}$ [$v$].

**3.** Solve the equations :

(a) $2\sqrt{x}=1-\sqrt{x}$ (b) $\dfrac{1}{x}+\dfrac{1}{2x}=\dfrac{3}{4}$

(c) $\sqrt{1-t}=\frac{1}{3}$ (d) $\sqrt{\dfrac{1+2m}{1-2m}}=0{\cdot}7$.

**4.** Express the letter in brackets in terms of the other letters :

(a) $D=\sqrt{\dfrac{3h}{2}}$ [$h$] (b) $A=p+\dfrac{prt}{100}$ [$p$]

(c) $T=2\pi\sqrt{\dfrac{l}{g}}$ [$g$] (d) $V=\pi r^2(h-\frac{1}{3}r)$ [$h$]

(e) $f=\dfrac{1}{2\pi\sqrt{LC}}$ [C] (f) $\sqrt{r^2-(r-h)^2}=y$ [$r$]

**5.** If $2s=a+b+c$, express $b+c-a$ in terms of $s$ and $a$ and prove that $\dfrac{(b+c-a)(c+a-b)}{(a+b+c)(a+b-c)}=\dfrac{(s-a)(s-b)}{s(s-c)}$.

**6.** From the equation $k+kmx=m-x$ express (a) $k$, (b) $m$, (c) $x$ in terms of the other letters.

7. If two shafts of diameters $d_1$ and $d_2$ with a belt of thickness $t$ round them make $n_1$ and $n_2$ revolutions per minute,

$$\frac{n_1}{n_2}=\frac{d_2+t}{d_1+t};$$

make $t$ the subject of this formula.

8. Make $\theta_m$ the subject of the formula $\dfrac{\theta_m - x}{\theta_m}=\dfrac{y}{\theta_1}$.

9. In calorimetry the following equation occurs:

$$\begin{aligned}Ms_1(t-32)+m_1s_2(t-32)+m_2(t-32)\\ =Ms_1(T-32)+m_1s_2(t_0-32)+m_2(t_0-32)\end{aligned}$$

Prove $$T=\frac{(m_1s_2+m_2)(t-t_0)}{Ms_1}+t.$$

10. If $\dfrac{1}{a}=\dfrac{2}{R}-\dfrac{1}{x}$ which of the following is the right value of $x$?

(a) $2a-R$ (b) $\dfrac{2a-R}{aR}$ (c) $\dfrac{aR}{2a-R}$ (d) $\dfrac{R}{2}-a$ (e) $\dfrac{aR}{R-2a}$.

11. If $\omega=2\pi f$ and $\omega L=\dfrac{1}{\omega C}$ express $f$ in terms of $\pi$, L and C.

12. Make W the subject of the formula $N=188\sqrt{\dfrac{EI}{Wl^4}}$.

13. If $T_c=M+\sqrt{M^2+T^2}$ express M in terms of T and $T_c$ [write the equation $T_c-M=\sqrt{M^2+T^2}$ and square both sides].

14. Make $\omega^2$ the subject of the formula $\dfrac{R_1}{R_2}=\dfrac{\sqrt{R_3^2+L_3^2\omega^2}}{\sqrt{R_4^2+L_4^2\omega^2}}$.

15. A point divides a line 12 in. long into two parts so that one part is $\frac{1}{2}$ in. longer than the other. Find the lengths of the parts.

16. A lens is in the form of a segment of a sphere. It is 0·2 in. thick and the radius of the plane face is 3 in. Find the radius of the curved surface.

17. A cyclist A is riding at 12 m.p.h. Another cyclist, B, starts $\frac{1}{4}$ mile behind A and rides at 15 m.p.h. How long will he take to overtake A?

18. Find two numbers whose sum is 65 so that the sum of a third of one number and a quarter of the other is 18.

**19.** A man buys 1000 oranges at 10 for 2*s*. 0*d*. and sells some at 60% profit and the remainder at 50%. How many must he sell at the higher price to make a profit of £5 12*s*. 0*d*.

**20.** Find what resistance must be put in parallel with a resistance of 24 ohms to produce a resistance of 8 ohms ?

**21.** How many pounds of glycerine of specific gravity 1·260 must be added to 20 gallons of water to make the specific gravity of the mixture 1·1 ? [1 gallon of water weighs 10 lbs.]

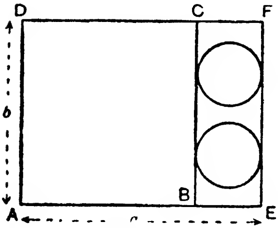



Fig. 1.

**22.** The figure shows a rectangular sheet of metal AEFD. The part ABCD is rolled into a cylinder of length AD and the circular discs cut from BEFC are used as ends of the cylinder. If the volume of the cylinder is V, show that,

$$a = 2(\pi + 1)\sqrt{\frac{V}{\pi b}}.$$

**23.** If a point on a thermometer is the same number on the Fahrenheit and Centigrade scales, find that number.

**24.** In a certain district electricity is charged for at two different rates, either (1) 5% of the rateable value of the house + 1*d*. a unit, or (2) 8*d*. a unit. For a house of rateable value £84 find the least number of units that must be consumed for it to pay the householder to go on the first rate. Find also the least number for a house of rateable value £*n*.

**25.** Two ships 500 sea miles apart are steaming towards each other at 30 and 20 knots. To what must the speed of the latter ship be increased in order that it may meet the former 2 hrs. earlier than it would have done ? Also find what speed is necessary if the ships are initially *n* miles apart.

**26.** A U tube with both ends open contains mercury to within 10 cm. of the ends of the tube. Water is poured into one arm until the water is 4 cm. below its end. Find how far the mercury rises in the other arm ? (sp. gr. mercury = 13·6.)

**27.** A car averaged 19 miles per gallon of petrol for 500 miles. How many miles per gallon must it average for the next 500 miles in order to increase the average over the whole 1000 miles to 21 miles per gallon ?

**28.** There are two rates of charging for electricity in a certain district : (*a*) ordinary rate :—light 8*d*. per unit, power 2*d*.. per

unit; (*b*) tariff rate :—5% of rateable value and 1*d*. per unit for light and power. If a householder has a house of rateable value £60 find the total cost of 100 units for lighting and 500 units for power, under both schemes. If he actually uses 80 units for lighting what is the least number of units he must use for power in order that it shall pay him to go on the tariff rate?

### Extended use of symbols. Units

In Part I symbols were used to represent numbers only, but, as we shall see below, the scope of a formula is greatly enlarged by supposing that each symbol in it represents not only the number which measures the quantity, but the unit in which the quantity is measured as well. Consider the formula $V=abc$, which gives the volume V cu. in. of a rectangular box of which the internal dimensions are $a$ in., $b$ in. and $c$ in. This formula is clearly true if the internal dimensions are $a$ cm., $b$ cm. and $c$ cm. and the volume is V cu. cm. In fact it is true whatever unit of length is used, provided the unit of volume is the volume of a cube whose edge is the unit of length. Because the formula is true whatever unit of length is used, it is useful to understand $a$, $b$, $c$ to represent not only numbers of inches, centimetres, etc., but to include the unit. For instance, if the dimensions of a box are 3 ft. by 2 ft. by $1\frac{1}{2}$ ft. we write $a=3$ ft., $b=2$ ft. and $c=1\frac{1}{2}$ ft., then

$$\begin{aligned} V &= 3\text{ ft.} \times 2\text{ ft.} \times 1\tfrac{1}{2}\text{ ft.} \\ &= (3 \times 2 \times 1\tfrac{1}{2})\text{ ft.} \times \text{ft.} \times \text{ft.} \\ &= 9\ (\text{ft.})^3, \end{aligned}$$

where $(\text{ft.})^3$ is written as short for ft. × ft. × ft. We see that $(\text{ft.})^3$ is a convenient shorthand for 1 cubic foot. It does not mean that we can multiply a foot by a foot; this is as impossible as multiplying an apple by an apple.

If we make $a$ the subject of the formula, we get:

$$a=\frac{V}{bc}.$$

Now suppose $V = 60$ cu. in., $b = 4$ in., $c = 5$ in. Then

$$a = \frac{60 \text{ (in.)}^3}{4 \text{ in.} \times 5 \text{ in.}} = \frac{60}{20} \frac{\text{(in.)}^3}{\text{(in.)}^2} = 3 \text{ in.}$$

This shows that the correct unit for the answer is obtained by treating the units as if they obeyed the index rules in the same way that numbers and symbols do.

If a car travels 200 ft. in 5 sec. its average speed is $\frac{200}{5}$ ft. per sec., i.e. 40 ft. per sec. We write this as $\frac{\overset{40}{\cancel{200}} \text{ ft.}}{\cancel{5} \text{ sec.}} = 40 \frac{\text{ft.}}{\text{sec.}}$ In the same way an acceleration of 3 ft. per sec. per sec. is written $3 \frac{\text{ft.}}{\text{sec.}^2}$.

*Example.*—The distance $s$ fallen by a body under gravity in time $t$ is given by $s = \frac{1}{2}gt^2$. Taking $g = 32$ ft. per sec. per sec., find the distance fallen in $\frac{1}{2}$ sec., putting the units in the formula.

Since
$$g = 32 \frac{\text{ft.}}{\text{sec.}^2},$$

$$s = \tfrac{1}{2} \cdot 32 \frac{\text{ft.}}{\text{sec.}^2} \cdot (\tfrac{1}{2} \text{ sec.})^2$$

$$= \frac{32}{2 \times 4} \frac{\text{ft.}}{\cancel{\text{sec.}^2}} \cdot \cancel{\text{sec.}^2}.$$

$$= 4 \text{ ft.}$$

When, from experience, we are sure of the unit of the answer it is unnecessary to put in the units in each line, as in the above example, but everyone should be able to check the units of an answer in this way.

*Example.*—Express a pressure of 1 lb. wt. per sq. in. in grams wt. per sq. cm.

$$1 \text{ lb. wt. per sq. in.} = \frac{1 \text{ lb. wt.}}{\text{in.}^2} = \frac{454 \text{ gm. wt.,}}{(2{\cdot}54 \text{ cm.})^2},$$

$$=\frac{454 \text{ gm. wt.}}{6{\cdot}45 \text{ cm.}^2}$$

$$=70{\cdot}4 \text{ gm. wt. per sq. cm.}$$

*Example.*—Young's modulus of elasticity for a metal bar is given by

$$E=\frac{4Wl^3}{bd^3y}$$

where $l$ = length, $b$ = breadth, $d$ = depth of the bar, and $y$ is the deflection of one end due to a load W when the other end is clamped horizontally.

If $b$, $d$, $y$, $l$ are measured in inches and W in lb. wt., in what unit is E measured ? Also construct a formula to give E in $\frac{\text{ton. wt.}}{\text{ft.}^2}$ when W is in cwt., $l$ in ft. and $b$, $d$, $y$ in inches.

Writing the units inches and lb. wt. alongside the symbols

$$E=\frac{4W \text{ lb. wt. } (l \text{ in.})^3}{(b \text{ in.})\,(d \text{ in.})^3\,(y \text{ in.})}=\frac{4Wl^3}{bd^3y}\,\frac{\text{lb. wt. (in.)}^3}{(\text{in.})^5}=\frac{4Wl^3}{bd^3y}\,\frac{\text{lb. wt.}}{(\text{in.})^2}.$$

This shows that the unit for E is 1 lb. wt. per sq. in.

If W is in cwt., $l$ in ft., $b$, $d$, $y$ in inches.

$$E=\frac{4W \text{ cwt. } (l \text{ ft.})^3}{bd^3y\,(\text{in.})^5}=\frac{4W\,.\,\frac{\text{ton wt.}}{2240}\,l^3(\text{ft.})^3}{bd^3y\left(\frac{\text{ft.}}{12}\right)^5}$$

$$\therefore\ E=\frac{4\times 12^5}{2240}\,\frac{Wl^3}{bd^3y}\,\frac{\text{ton wt.}}{(\text{ft.})^2}.$$

Thus we can say that for this set of mixed units :

$$E=\frac{4\times 12^5}{2240}\,\frac{Wl^3}{bd^3y}.$$

*Example.*—Ohm's Law E = RI, which relates the electromotive force E, resistance R and the current I in a wire, is true when E, R, I are measured in the practical units, volt, ohm and

ampère, and when they are measured in C.G.S. units. Assuming 1 ampère $=\frac{1}{10}$th C.G.S. unit of current, 1 volt $=10^8$ C.G.S. units of electromotive force, find the number of C.G.S. units of resistance in 1 ohm.

Since E = RI is true in practical units

$$1 \text{ volt} = 1 \text{ ohm} \times 1 \text{ ampère}$$

$$\therefore 1 \text{ ohm} = \frac{1 \text{ volt}}{1 \text{ ampère}}$$

$$= \frac{10^8 \text{ C.G.S. units of E.M.F.}}{\frac{1}{10} \text{ C.G.S. unit of current}}$$

$$= 10^9 \text{ C.G.S. units of resistance}$$

(since 1 C.G.S. unit of E.M.F. = 1 C.G.S. unit of resistance × 1 C.G.S. unit of current).

## Exercise IV

**1.** If $t$ is a time in seconds, $s$ a distance in feet, $v$ a velocity in ft. per sec., $f$ an acceleration in ft. per sec. per sec., find the units of each of the following and state what kind of a quantity each is:

(a) $ft$  (b) $\frac{1}{2}ft^2$  (c) $\sqrt{fs}$  (d) $\frac{v^2}{f}$.

**2.** If R is a resistance in ohms, $i$ a current in ampères, V a potential difference in volts, find the units of each of the following in their simplest form, assuming 1 volt = 1 ohm × 1 ampère and 1 watt = 1 volt × 1 ampère.

(a) $\frac{V}{R}$  (b) $Ri^2$  (c) $\frac{V^2}{R}$.

**3.** If $u$ and $v$ are velocities in feet per second, $s$ a distance in feet, and $t$ a time in seconds, find in which of the formulæ

(a) $v^2 = u^2 + 2ft$  (b) $v^2 = u^2 + 2fs$

the units of all the terms are the same. Which formula do you think is correct?

**4.** If W is in lb. wt., $v$ in ft./sec., $g$ in ft./sec.$^2$, in what unit is kinetic energy $\frac{1}{2}\frac{Wv^2}{g}$ measured?

5. The weight of a mass of one pound gives it an acceleration of nearly 32·19 ft./sec.$^2$ in London. What is the relation between 1 lb. wt., 1 lb. mass and 1 ft./sec.$^2$ ?

6. The absolute unit of force (a poundal) is the force which gives the standard mass 1 lb. an acceleration of 1 ft./sec.$^2$ What is the absolute unit in terms of lb. mass, ft. and sec. ? In the C.G.S. system the absolute unit is the dyne, a force which gives a mass of 1 gram an acceleration of 1 cm./sec.$^2$ What is the relation between a dyne, a gram, a centimetre and a second ? What is the relation between the weight of a kilogram and a dyne ?

7. Taking $g = 32{\cdot}19$ ft./sec.$^2$, 1 lb. mass $= 453{\cdot}6$ grams, 1 ft. $= 30{\cdot}48$ cm. and 1 dyne $=$ 1 gram. cm./sec.$^2$, express a force of 1 lb. wt. in dynes.

8. The formula for the tension of a wire when it is stretched a length $x$ is $T = (EAx)/l$, where A is the area of the cross-section of the wire, $l$ is the length of the wire and E is Young's Modulus of elasticity for the material of which the wire is made. If $l$ and $x$ are in inches, A is in square inches and T in lb. wt., in what units is E measured ?

9. If a body moves in a circle of radius $a$ with velocity $v$ it has an acceleration $\dfrac{v^2}{a}$ towards the centre. By taking $v$ in ft./sec. and $a$ in ft. show that $\dfrac{v^2}{a}$ has the same units as an acceleration.

10. If the potential difference between the plates of a condenser having a capacity C is V when the charge on a plate is Q, $Q = CV$, when C is in farads, V in volts and Q in coulombs. Taking R to be a resistance in ohms, $i$ a current in ampères, and assuming 1 ampère $=$ 1 coulomb/sec. show that:

(a) $\dfrac{Q^2}{Ct}$ is in watts. (b) $\dfrac{CV}{i}$ is a time in seconds.

(c) $RCi$ is a charge in coulombs.

## Functions

The current produced in a given wire of resistance 10 ohms by an electromotive force E volts is $\dfrac{E}{10}$ ampères. Because the current behaves or functions according to this relationship with E, it is called a function of the voltage E. If the current

is denoted by the letter $i$, then the formula $i = \frac{E}{10}$ expresses the functional relationship of $i$ with E.

In general, any quantity, whose value depends on the value of a variable quantity $x$, is called a function of $x$. $\frac{1}{2}x^2$, $3 \sin^2 t$, $2^n$ are functions of $x$, $t$ and $n$ respectively. Functions like $x^{\frac{1}{2}}$, $(1-p^2)^4$, $\frac{2r+1}{2r-1}$ are called algebraic functions; functions formed by using the trigonometrical ratios such as $\tan \theta°$, $3 \cos$ ($2t$ radians), $\sin^2 \alpha$, are called trigonometric functions; functions like $2^{-x}$, $(1{\cdot}83)^{3x}$ are called exponential functions, because the index or exponent is the variable part in them. An equation relating two variable quantities tells us what function one quantity is of the other; for example, $y = 5x^2$, $C = \frac{5}{9}(F - 32)$, $R = \frac{r+3}{r-3}$, show what function $y$ is of $x$, C of F and R of $r$ respectively. To see what function one quantity is of another it may be necessary to change the subject of the formula. For instance, to find what function F is of C from the formula $C = \frac{5}{9}(F - 32)$ we put the equation in the form $F = 32 + \frac{9}{5}C$.

A quantity may be a function of more than one other variable quantity. For instance, the volume of a cylinder

GRAPH OF THE FUNCTION $x^2$

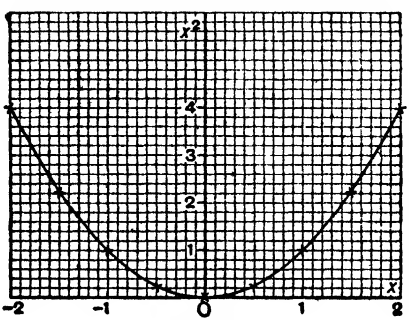



FIG. 2.

depends on both its height, $h$, and its radius, $r$; the volume is $\pi r^2 h$, and it is a function of both $r$ and $h$. The deflection of the middle point of a beam supported at its ends at the same level is a function of (1) its length, (2) its breadth, (3) its depth, (4) its weight per unit length, (5) its elasticity, making five variable quantities in all.

The graph of the equation $y = x^2$ may also be considered as the graph of the function $x^2$, and when we are thinking of it in this way we label the vertical axis "$x^2$" instead of $y$, as in Fig. 2.

## Exercise V

Express:

**1.** The electromotive force between two points in a wire of resistance 6 ohms as a function of the current $i$ ampères.

**2.** The area of a square as a function of the length $x$ of a diagonal.

**3.** The distance dropped by a body under gravity as a function of the time taken, $t$ sec. (take $g = 32$ ft./sec.$^2$).

**4.** The area of a circular ring of width 1 in. as a function of the radius $r$ in. of the inside of the ring.

**5.** The telephone bill of a subscriber as a function of the number of calls, $n$, in a quarter, if the charge for a telephone is £2 per quarter, the first 50 calls are free, and the charge for the remainder is 1d. a call.

**6.** The volume of a cube as a function of the area A of a face.

**7.** If $y = \dfrac{1}{2x} - 3$ express $x$ as a function of $y$.

**8.** If $m = \dfrac{n+1}{n-1}$ show that $n$ is the same form of function of $m$ as $m$ is of $n$.

**9.** If $y = x^2 - 1$ and $x = 1 - \frac{1}{2}u$, express $y$ as a function of $u$.

**10.** Write down the relationship between the area A of a circle and its radius $r$. What function is $r$ of A?

**11.** The pressure $p$, the volume $v$ and the temperature $t$ of a gas are related by the equation $pv = k(273 + t)$ where $k$ is a fixed number. Express $t$ as a function of $p$ and $v$.

12. Express the length of the hypotenuse of a right-angled triangle as a function of the lengths $a$ and $b$ of the other two sides.

13. If the height of a cylinder is twice its radius, express its volume as a function of (1) its height, (2) its radius.

14. In a triangle ABC the angle B is 90°. If AB and BC have lengths $a$ and $b$, express the perpendicular distance of B from AC as a function of $a$ and $b$.

15. The impedance Z of a certain electric circuit is given by $Z = \dfrac{1}{\sqrt{\dfrac{1}{R^2} + \omega^2 C^2}}$. Express $\omega$ as a function of Z, R and C.

16. If $xy + 3x = 4y + 5$, express $(a)$ $x$ as a function of $y$, $(b)$ $y$ as a function of $x$.

17. If $\dfrac{k}{v} - \dfrac{1}{u} = \dfrac{k-1}{r}$, express $k$ as a function of the other letters.

## Simultaneous equations

The following example shows the two methods of solving simultaneously equations of the first degree.

*Example.*—Solve the equations :

$$4x - 3y = 7 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$6x - 7y = -2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

1st Method.—Eliminate a letter, say $x$, by multiplying (1) and (2) by numbers which will make the coefficients of $x$ the same in both equations. These numbers can be 6 and 4, which make the coefficients of $x$ both 24, but since the L.C.M. of 6 and 4 is 12 it will do if (1) is multiplied by $\frac{12}{4} = 3$, and (2) by $\frac{12}{6} = 2$. Then :

$$12x - 9y = 21$$
$$12x - 14y = -4$$

Subtracting $\qquad 5y = 25$

$$\therefore y = 5$$

Substituting this value in (1) :

$$4x = 3y + 7 = 15 + 7 = 22$$
$$\therefore x = 5\tfrac{1}{2}.$$

2nd Method.—Make $x$ the subject of equation (1):

$$4x = 3y + 7$$

$$\therefore\ x = \frac{3y+7}{4}.$$

Substitute this value in equation (2):

$$6\left(\frac{3y+7}{4}\right) - 7y = -2$$

$$\therefore\ \frac{3(3y+7)}{2} - 7y = -2$$

$$\therefore\ 9y + 21 - 14y = -4$$

$$\therefore\ -5y = -25$$

$$\therefore\ y = 5,$$

Whence $\qquad x = 5\frac{1}{2}$ as before.

When the coefficients of $x$ and $y$ contain three or more figures it is better to make the coefficients of $x$ equal in the equations by dividing each equation by the coefficient of $x$ in it.

*Example.*—Solve the equations:

$$4{\cdot}97x + 63{\cdot}1y = 26{\cdot}9, \qquad 27{\cdot}3x - 15{\cdot}2y = 65{\cdot}4.$$

Dividing by 4·97 and 27·3 respectively:

$$x + 12{\cdot}70y = 5{\cdot}412$$

$$x - 0{\cdot}557y = 2{\cdot}396$$

$$\therefore\ \text{substracting, } 13{\cdot}26y = 3{\cdot}016$$

$$\therefore\ y = \frac{3{\cdot}016}{13{\cdot}26} = 0{\cdot}2275.$$

$$\therefore\ x = 5{\cdot}412 - 12{\cdot}70y = 5{\cdot}412 - 12{\cdot}70 \times 0{\cdot}2275 = 2{\cdot}521.$$

Ans. $x = 2{\cdot}52$, $y = 0{\cdot}228$.

Either of the methods used above may be used to eliminate a letter from two formulæ.

*Example.*—If two weights, W lb. wt. and $w$ lb. wt. are connected by a string which passes over a light smoothly-running

pulley, their acceleration $f\,\dfrac{\text{ft.}}{\text{sec.}^2}$ and the tension of the string T lb. wt. are given by

$$\frac{Wf}{g} = W - T \text{ and } \frac{wf}{g} = T - w.$$

Find T in terms of W, $w$ and $g$.

To eliminate $f$ multiply the first equation by $w$ and the second by W.

$$\frac{Wwf}{g} = Ww - Tw$$

$$\frac{Wwf}{g} = TW - Ww$$

Subtracting $\qquad 0 = 2Ww - Tw - TW$

$$\therefore\ TW + Tw = 2Ww$$

$$\therefore\ T(W + w) = 2Ww$$

$$\therefore\ T = \frac{2Ww}{W + w}.$$

On the other hand, if we use the substitution method, we make $f$ the subject of the first formula $\dfrac{Wf}{g} = W - T$, by multiplying by $\dfrac{g}{W}$. This gives

$$f = \frac{g}{W}(W - T)$$

Substituting this value in the second formula,

$$\frac{w}{g} \cdot \frac{g}{W}(W - T) = T - w$$

Multiplying by W,

$$w(W - T) = W(T - w)$$

$$\therefore\ wW - wT = WT - Ww$$

Whence, as above, $\qquad T = \dfrac{2Ww}{W + w}.$

## Exercise VI

Solve the simultaneous equations :

1. $3x + 6y = 11$
   $14x - y = 3.$

2. $2x - 3y = 43$
   $3x + 5y = 18.$

3. $26x + 8y = 9$
   $2y - 6x = 2.$

4. $9x + 14y = 5$
   $12x + 21y = 7.$

Find the roots of the following equations to 3 significant figures.

5. $0{\cdot}05x + 0{\cdot}2y = 4{\cdot}5$
   $0{\cdot}3x + 0{\cdot}04y = 5{\cdot}5$

6. $14{\cdot}71x - 1{\cdot}96y = 20{\cdot}43$
   $2{\cdot}95x + 8{\cdot}24y = 16{\cdot}6.$

7. $1023x - 47y = 515$
   $147x + 212y = -100.$

8. $8{\cdot}14x + 0{\cdot}92y = 10{\cdot}07$
   $2{\cdot}93x + 12{\cdot}12y = 18{\cdot}95.$

9. Find $p$ and $q$ if $p - \frac{1}{2}q = \frac{1}{3}p + q = 7.$

10. The effort P lb. wt. required to raise a load W lb. wt. by means of a differential pulley block is given by $P = aW + b$. It is found in an experiment that $P = 1{\cdot}71$ when $W = 15$, and $P = 4{\cdot}36$ when $W = 75$. Find $a$ and $b$.

11. Calculate the values of $m$ and $c$, if $y = mx + c$ given that when $x = 6$, $y = 10$, and when $x = 15$, $y = 37$.

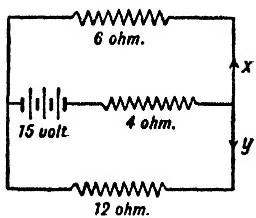



Fig. 3.

12. If, in the circuit shown, the currents in the two parts are $x$ and $y$ ampères,

$$4(x + y) + 6x = 15$$

and $$6x - 12y = 0.$$

Find $x$ and $y$.

13. If in the circuit in Question 12 the battery has a voltage E and the resistances are R, $r$ and $s$ instead of 4, 6 and 12,

$$R(x + y) + rx = E$$
$$rx - sy = 0$$

Find $x$ and $y$ in terms of the other letters.

14. Fig. 4 shows the forces on a 20-lb. wt. when the weight is just being pushed up the plane by a horizontal force P lb. wt., and the coefficient of friction is $0{\cdot}35$. By resolving parallel

and perpendicular to the plane, the following equations are obtained :

$$P \cos 30^\circ = 0{\cdot}35N + 20 \sin 30^\circ$$
$$P \sin 30^\circ + 20 \cos 30^\circ = N.$$

Solve these equations for P and N.

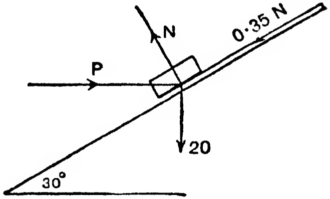



FIG. 4.

**15.** In a problem in mechanics the following equations occur :

$$\frac{\sqrt{3}}{2}\,T_1 - \tfrac{1}{2}T_2 = 50$$

$$\tfrac{1}{2}\,T_1 + \frac{\sqrt{3}}{2}\,T_2 = 100.$$

Find $T_1$ and $T_2$.

**16.** If $e = k\text{DG} = kd(\text{G}+\text{R})$, find $k$ in terms of $e$, D, $d$ and R, but not G.

**17.** In calculating the strength of a reinforced concrete column the following equations occur: $f = m\text{F} = \dfrac{\text{L}-\text{L}_c}{a}$, $\text{F} = \dfrac{\text{L}_c}{\text{A}}$; prove that $f = \dfrac{m\text{L}}{am+\text{A}}$.

**18.** If $f = \dfrac{1}{2\pi\sqrt{\text{LC}}}$ prove that $2\pi f\text{L}_1 - \dfrac{1}{2\pi f\text{C}_1} = \dfrac{\text{L}_1\text{C}_1 - \text{LC}}{\text{C}_1\sqrt{\text{LC}}}$.

**19.** If $\omega = 2\pi f$ and $\omega\text{L} = \dfrac{1}{\omega\text{C}}$, express $f$ in terms of $\pi$, L and C.

**20.** If $\text{B} = \dfrac{4\pi\text{IN}}{10l}$ and $\text{L} = \dfrac{4\pi\text{N}^2a10^{-9}}{l}$, prove that $\tfrac{1}{2}\text{LI}^2 = \dfrac{\text{B}^2al10^{-7}}{8\pi}$.

**21.** If $\text{A} = akk' + k + k'$, $\text{B} = ak' + 1$, $\text{C} = ak + 1$, express $a$, $k$ and $k'$ in terms of A, B and C.

22. If $p_x = \dfrac{16M}{\pi d^3}$, $p_y = \dfrac{16T}{\pi d^3}$ and $p_1 = \frac{1}{2}\{p_x + \sqrt{p_x^2 + p_y^2}\}$, prove that $p_1 = \dfrac{8}{\pi d^3}\{M + \sqrt{M^2 + T^2}\}$.

23. If $V^2T_1^2 = d^2 + a_1^2$ and $V^2T_2^2 = d^2 + a_2^2$, express $V^2$ and $d^2$ in terms of $T_1$, $T_2$, $a_1$ and $a_2$.

24. If $\dfrac{x}{h} = \dfrac{1}{\sqrt{3}d}$, $\dfrac{1}{d} = \dfrac{y}{\sqrt{3}}$ and $hy - x = 1$, express $h$ in terms of $d$.

## Graphs

To plot the graph of an equation such as

$$y = 20 - 5x^2 + \frac{40}{x+5}$$

for a given range of values of $x$, say $x = -3$ to $x = 4$, make a table of the values of each term of $y$ for each of the selected values of $x$ and then add the columns to obtain the values of $y$.

| $x$ .. | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|
| 20 .. | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 |
| $-5x^2$ .. | $-45$ | $-20$ | $-5$ | 0 | $-5$ | $-20$ | $-45$ | $-80$ |
| $\dfrac{40}{x+5}$ .. | 20 | 13·3 | 10 | 8 | 6·7 | 5·7 | 5 | 4·4 |
| Sum $= y$ | $-5$ | 13·3 | 25 | 28 | 21·7 | 5·7 | $-20$ | $-55·6$ |

A graph is plotted from this table in Fig. 5. Such a graph can be used for the following purposes :

(*a*) to read off the value of $y$ for any value of $x$ between $-3$ and 4, or to read off the values of $x$ between $-3$ and 4, which make $y$ have any value between $-55·6$ and $+28·1$. This is called interpolation. For example, we find that when $x = 1·5$, $y \simeq 14·75$ at A, and when $y = 10$, $x \simeq -2·2$ or 1·8 at B and C.

(*b*) to find any maximum or minimum values $y$ may have in the given range of values of $x$.

In Fig. 5, $y$ has a maximum value 28·1 at $x \simeq -0·2$ at D.

GRAPH OF $y=20-5x^2+\frac{40}{x+5}$

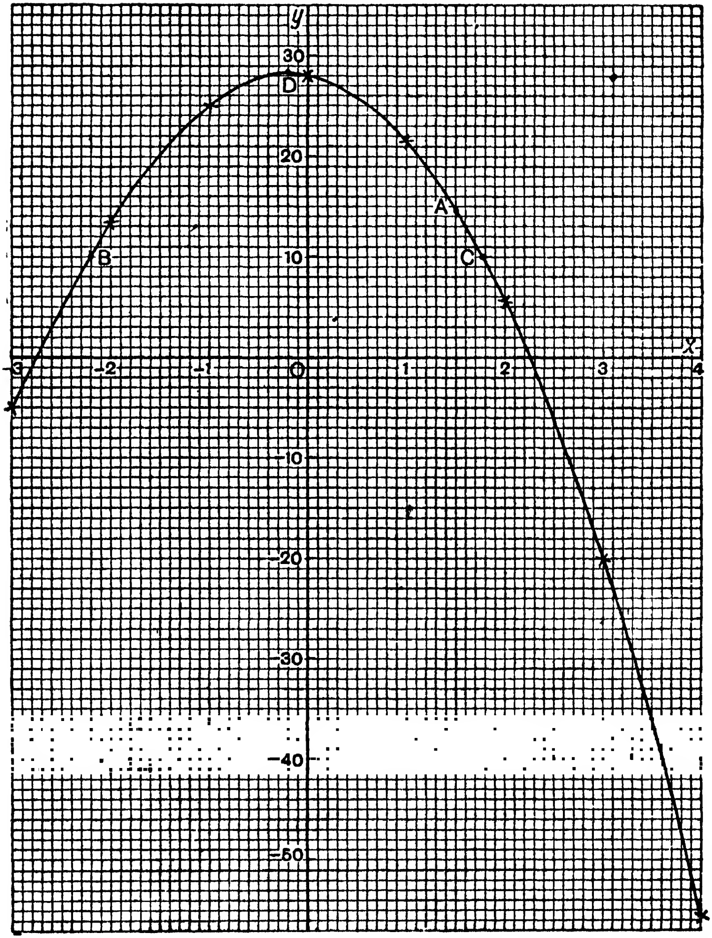



Fig. 5.

To obtain any one of these values more accurately the part of the graph near the corresponding point has to be plotted to a much larger scale. For instance, to find the maximum value of $y$ its values at $x = -0{\cdot}3$, $-0{\cdot}2$ and $-0{\cdot}1$ are tabulated in addition to the value at $x=0$, which has already been found.

| | | | | | |
|---|---|---|---|---|---|
| $x$ | .. | $-0{\cdot}3$ | $-0{\cdot}2$ | $-0{\cdot}1$ | $0$ |
| $20$ | .. | $20$ | $20$ | $20$ | |
| $-5x^2$ | .. | $-0{\cdot}45$ | $-0{\cdot}20$ | $-0{\cdot}05$ | |
| $\dfrac{40}{x+5}$ | .. | $8{\cdot}511$ | $8{\cdot}333$ | $8{\cdot}163$ | |
| $y$ | .. | $28{\cdot}061$ | $28{\cdot}133$ | $28{\cdot}113$ | $28$ |

A graph is plotted from this table in Fig. 6, and it shows that $y$ has a maximum value $28{\cdot}14$, at $x \simeq -0{\cdot}17$. It should be noted that the maximum value can be found far more accurately than the corresponding value of $x$; moreover, $y \simeq 28$ from $x = -0{\cdot}3$ to $x=0$ so that any value of $x$ in this range makes $y$ very nearly equal to its maximum.

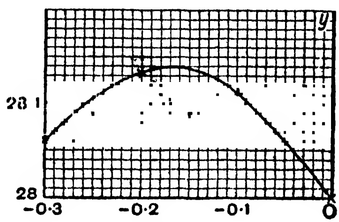


FIG. 6.

*Example.*—The time of a small oscillation, T sec., of a rod of length 120 cm. swinging in a vertical plane about an axis at $h$ cm. from its middle point is given by

$$T = 2\pi\sqrt{\frac{h^2+1200}{gh}},$$

where $g=981$. Draw a graph to show the variation of T with $h$ from $h=0$ to $h=60$. Find from it the value of $h$ for which $T=2$, and the minimum value of T.

Putting $g=981$,

$$T = \frac{2\pi}{\sqrt{981}}\sqrt{\frac{h^2+1200}{h}} \simeq 0{\cdot}2\sqrt{h+\frac{1200}{h}}.$$

GRAPH TO SHOW THE TIME OF OSCILLATION OF A ROD

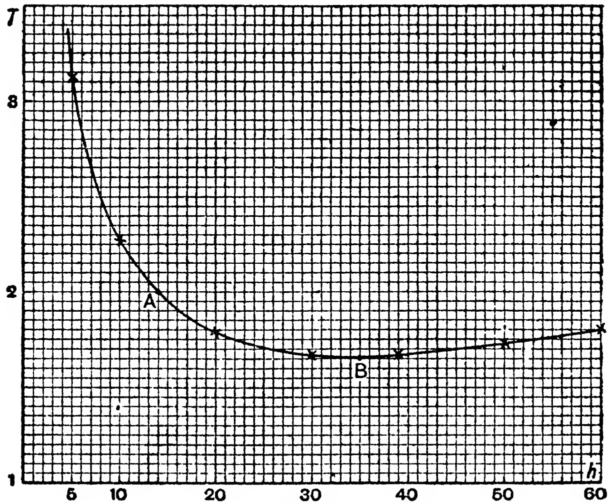



FIG. 7.

The table below shows how T is calculated from $h$. The values of $h$ are taken at intervals of 10 cm., except the first value, which is taken at $h=5$ because T is infinite when $h$ is zero.

| $h$ | 5 | 10 | 20 | 30 | 40 | 50 | 60 |
|---|---|---|---|---|---|---|---|
| $\frac{1200}{h}$ | 240 | 120 | 60 | 40 | 30 | 24 | 20 |
| $h+\frac{1200}{h}$ | 245 | 130 | 80 | 70 | 70 | 74 | 80 |
| $\sqrt{h+\frac{1200}{h}}$ | 15·65 | 11·40 | 8·94 | 8·37 | 8·37 | 8·60 | 8·94 |
| T | 3·13 | 2·28 | 1·79 | 1·67 | 1·67 | 1·72 | 1·79 |

Fig. 7 shows the graph of T against $h$. It shows that $T=2$ when $h \simeq 14$, and that the minimum value of T is approximately 1·66 and it occurs at $h \simeq 35$.

**Straight line graphs. Gradient of a line**

Consider the graph of $y=2x+3$. The table below gives the values of $y$ from $x=-3$ to $x=3$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ .. .. | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
| $y$ .. .. | $-3$ | $-1$ | $1$ | $3$ | $5$ | $7$ | $9$ |

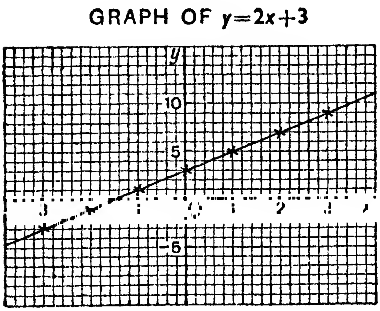



FIG. 8.

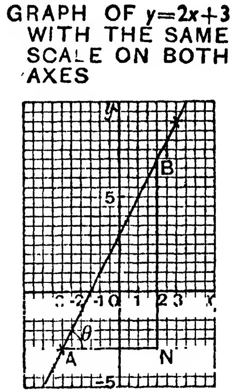



FIG. 9.

Thus $y$ increases by 2 whenever $x$ increases by 1. Hence the graph of the equation is a straight line as shown in Fig. 8. $y=2x+3$ is called the equation of the line. The ratio (increase of $y$) : (increase of $x$) is 2 whatever pair of points is taken on the line. For instance, from the table above, two points on the line are A given by $x=-3$, $y=-3$, called the point $(-3, -3)$, and B given by $x=2$, $y=7$, called the point $(2, 7)$. From A to B the increase of $y$ is $7-(-3)=10$, and the increase of $x$ is $2-(-3)=5$.

$$\therefore \frac{\text{increase of } y}{\text{increase of } x}=\frac{10}{5}=2.$$

In mathematics this ratio is called the *gradient of the line.*

Fig. 9 shows the graph of $y=2x+3$, using the same scale on both axes.

*When the scales on the two axes are the same* the gradient is the ratio of the actual lengths of the vertical rise NB to the horizontal distance AN. If the line makes an angle $\theta$ with the horizontal, $\tan\theta = \frac{NB}{AN}$, and so :

*Gradient of the line = tan $\theta$, when the scales on the axes are the same.*

In Fig. 9, $\tan\theta = 2$, and hence $\theta \simeq 63° 26'$.

In general the graph of $y = ax + b$, where $a$ and $b$ are given numbers, is a straight line of gradient $a$. Since $y = b$ when $x = 0$, the constant $b$ is the intercept on O$y$ measured from O to the point where the line cuts O$y$. If the line cuts the $y$ axis below O, then $b$ is negative.

GRAPH OF $y = -\frac{1}{2}x + 1$

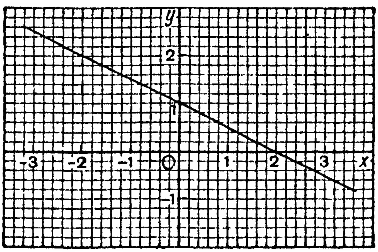



Fig. 10.

If $a$ is negative the gradient of the line is still $a$. For instance, in the equation $y = -\frac{1}{2}x + 1$, when $x$ increases by 1, $y$ decreases by $\frac{1}{2}$, so we say that $y$ increases by $-\frac{1}{2}$.

$$\therefore \frac{\text{increase of } y}{\text{increase of } x} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}.$$

Hence the gradient of the line is $-\frac{1}{2}$. The graph of this line is shown in Fig. 10. It is clear that lines with a positive gradient slope upwards to the right and lines with a negative gradient slope downwards to the right.

Any equation like $5x+4y=8$, in which the terms in $x$ and $y$ are of the first degree, has a straight line graph, for it can be written

$$4y=8-5x$$

$$\text{i.e. } y=2-1{\cdot}25x,$$

which is the straight line of gradient $-1{\cdot}25$ shown in Fig. 11.

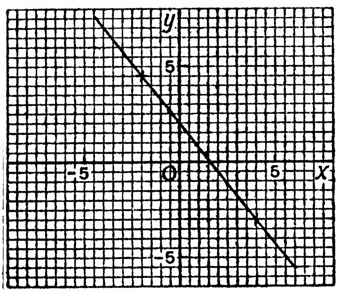



FIG. 11.

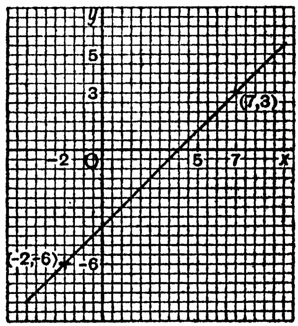



FIG. 12.

## Equation of the line through two points

If the graph of $y=ax+b$ passes through the two points $(-2, -6)$, $(7, 3)$ [Fig. 12], then $y=-6$ when $x=-2$ and $y=3$ when $x=7$.

$$\therefore\ -6=-2a+b$$

and $$3=7a+b$$

Subtracting $$-9=-9a$$

$$\therefore\ a=1 \text{ and } b=3-7a=3-7=-4.$$

Hence the required equation is $y=x-4$.

## Exercise VII

Draw the graphs of the following equations for the given range of values and use the graphs to answer the questions :

**1.** $y=12x^2-18x-7$ from $x=-2$ to 4. Find the minimum value of $y$ and the values of $x$ which make $y=10$.

**2.** $R=1+6v-2v^3$ from $v=-2$ to 2. Find the maximum and minimum values of R and the values of $v$ which make $R=0$.

**3.** $y=x^3+4x-8$ from $x=-2$ to 3. Find to 3 sig. fig. the values of $x$ which make $y=0$ and $y=-10$ respectively.

**4.** $t=m^2+\frac{7}{m}$ from $m=0$ to 4. Find to 3 sig. fig. the minimum value of $t$ and the value of $m$ for which $t$ is a minimum.

**5.** $z=y^3-40y$ from $y=-5$ to 5. Find the maximum and minimum values of $z$. If P is a point on the graph and PQ is bisected by the origin, prove that Q is also a point on the graph.

**6.** $y=\frac{10}{x^2+4}-x$ from $-2$ to 3. Find to 3 sig. fig. the values of $x$ which make $y=0$ and 2 respectively.

**7.** $p=8\sqrt{r}-3r$ from $r=0$ to 5. Find the maximum value of $p$ to 3 sig. fig. Has $p$ a graph for negative values of $r$ ?

**8.** $y=4x^3-3x^2-14x+4$ from $x=-2$ to 3. Find the values of $x$ which make $y=0$, giving the larger positive one to 3 sig. fig.

**9.** $y=\frac{2}{3}\sqrt{9-x^2}$ and $y=-\frac{2}{3}\sqrt{9-x^2}$ from $x=-3$ to $+3$. Show that these two graphs together make up the graph of $\frac{x^2}{9}+\frac{y^2}{4}=1$.

**10.** $y=\frac{1}{x}$ from $x=-4$ to $+4$, taking values of $x$ at intervals of $\frac{1}{4}$. Calculate the values of $y$ when $x=-0{\cdot}01$, $-0{\cdot}001$, $0{\cdot}001$, $0{\cdot}01$. What happens to the graph as $x$ increases through the value 0 ?

**11.** $y=\sin\theta^\circ+2\cos\theta^\circ$ from $\theta=0$ to $\theta=90$. Find the maximum value of $y$ to 3 sig. fig., and the value of $\theta$ (to the nearest degree) which makes $y=1{\cdot}5$.

Plot the graphs of the following functions :

**12.** $2x^3$ from $x=-3$ to 3. **13.** $\frac{1}{x^2}$ from $x=-3$ to 3.

**14.** $t(6-t)$ from $t=-2$ to 8.

**15.** $\sin(\theta \text{ radians})$ from $\theta=0$ to $2\pi$.

**16.** $4m^3-16m^2$ from $m=-2$ to 2.

**17.** $2^x$ from $x=-3$ to 3.

**18.** $\log_{10}x$ from $x=\frac{1}{4}$ to 4.

**19.** $\tan x°$ from $x=0$ to 180.

Draw the straight lines which are the graphs of the following equations and state the gradient of each line and the intercept it makes on the vertical ($y$, C or V) axis.

**20.** $y=5-3x$.

**21.** $2y+x=8$.

**22.** $3y-4x=7$.

**23.** $5y=10x-12$.

**24.** $C=\frac{5}{9}(F-32)$.

**25.** $V=200-0{\cdot}8I$.

Find the equations of the straight lines joining the following pairs of points:

**26.** (2, 3), (8, 11).

**27.** (6, 4), (1, 12).

**28.** (−2, 5), (3, 0).

**29.** When a beam is clamped horizontally at one end the deflection $y$ of the other end, which is distant $l$ from the clamp, is given by $y=\frac{Wl^3}{3EI}$, where W is the load hung from that end. Draw a graph of $y$ from $l=10$ to $l=100$ when $W=2$, $E=20\times10^6$, $I=0{\cdot}0012$.

**30.** Two wires containing resistances 10 ohms and R ohms are placed in parallel and a voltage V is applied to their ends. If the current through the resistance R ohms is 15 ampères, $V=\frac{150R}{R+10}$. Draw a graph of V against R from $R=1$ to $R=50$.

**31.** The formula $H=\frac{7{\cdot}35V}{V+10}-0{\cdot}13$ has been used to find the percentage, H, of hydrogen in coal when the percentage of volatile matter is V. V ranges from 10 to 50. Draw a graph of H against V for this range of values. From it find the value of V which makes $H=5$ and check your answer by calculation.

**32.** Draw the graphs of the equations $\frac{x}{3}+\frac{y}{4}=1$, and $y=x^2-1$ on the same axes and find the co-ordinates of the points where they intersect.

**33.** For an L.M.S. train of one locomotive and twelve coaches the resistance $r$ lb. wt. per ton at a speed of V m.p.h. is given by

$$r=5{\cdot}25+0{\cdot}0513V+0{\cdot}00162V^2.$$

Draw a graph of $r$ from $V=0$ to $V=80$. Determine at what speed $r=10$ and at what speed $r$ is double its value at 30 m.p.h.

34. For temperatures between 700° C. and 3000° C. the resistivity of tungsten, $r$ microhms per cubic cm. at $t$° C., is given by $r = 4{\cdot}053 + 0{\cdot}0275t + 0{\cdot}00000175t^2$. Draw a graph of $r$ against $t$ from $t = 700$ to $t = 3000$ and find what value of $t$ makes $r = 47$.

35. A load of 200 lb. wt. is suspended by two equal ropes, length $l$ ft., from two points 6 ft. apart at the same level. The tension T lb. wt. in each rope is then given by $T = \frac{100l}{\sqrt{l^2 - 9}}$. Plot a graph of T from $l = 4$ to $l = 10$. For what value of $l$ is T equal to 150 ?

36. For a certain material the Brinell hardness number H is given in terms of the diameter of indentation $d$ by the formula $H = \frac{W}{\frac{\pi D}{2}(D - \sqrt{D^2 - d^2})}$. It can be shown that this formula is approximately the same as $H = \frac{W}{\pi}\left(\frac{4}{d^2} - \frac{1}{D^2}\right)$ when $d/D$ is small. Draw graphs of both these formulæ from $d = 1$ to 9 when $W = 3000$ and $D = 10$.

37. When a square coil of 4 turns, each side 40 cm., carries a current of 12 ampères per turn, the magnetic force H at a point on the diagonal at $x$ cm. from the centre is given by

$$H = \frac{19{\cdot}2}{800 - x^2}(20\sqrt{2} + \sqrt{800 + x^2}).$$

Draw a graph to show how H varies as $x$ varies from $-25$ to 25.

# CHAPTER II

# FACTORS AND FRACTIONS

## Factors

The first step in factorizing an expression is to find the highest common factor of its terms.

*Example.*—Factorize $4l^2x^2 - 2lx^3$.

H.C.F. of $4l^2x^2$ and $2lx^3$ is $2lx^2$.

$$\therefore\ 4l^2x^2 - 2lx^3 = 2lx^2 \times 2l - 2lx^2 \times x$$
$$= 2lx^2(2l - x).$$

*Example.*—Factorize $(a+1)(2a+3)-(a+1)(b+3)$.

$(a+1)$ is a factor of both products and hence of the whole expression.

$$\therefore \text{ expression} = (a+1)\{(2a+3)-(b+3)\}$$
$$= (a+1)\{2a+3-b-3\}$$
$$= (a+1)(2a-b).$$

In many cases an expression is the product of factors, although its terms have no common factor. Such an expression can often be factorized by grouping its terms.

*Example.*—Factorize $a^2-2lm-2al+am$.

$$\text{Expression} = a^2-2al-2lm+am$$
$$= a(a-2l)+m(a-2l)$$
$$= (a+m)(a-2l).$$

**Factors of $a^2-b^2$.**

We have seen on p. 7 that

$$a^2-b^2=(a+b)(a-b).$$

This formula can be used to factorize the difference of any two squares.

*Example.*—Factorize $9x^2-4y^2$.

$$9x^2-4y^2=(3x)^2-(2y)^2$$
$$=(3x+2y)(3x-2y)$$

by putting $a=3x$ and $b=2y$ in the formula above.

*Example.*—Factorize $l^4-m^4$.

$$l^4-m^4=(l^2)^2-(m^2)^2=(l^2+m^2)(l^2-m^2)=(l^2+m^2)(l+m)(l-m).$$

**The calculation of $a^2-b^2$ when $a$ is very nearly equal to $b$.**

Suppose $a$ and $b$ are 12·00544 and 12 respectively. Then all we can get using four figure tables of squares is:

$$a^2-b^2=(12{\cdot}01)^2-12^2=144{\cdot}2-144=0{\cdot}2.$$

This answer is not even correct to the one significant figure 2, and so we proceed as follows :

$$\begin{aligned} a^2-b^2 &= (a-b)(a+b) \\ &= (12{\cdot}00544-12)(12{\cdot}00544+12) \\ &= 0{\cdot}00544 \times 24{\cdot}00544 \\ &\simeq 0{\cdot}00544 \times 24 \\ &\simeq 0{\cdot}131. \end{aligned}$$

The error in taking 24 instead of 24·00544 is less than 1 in 4000 and so will not affect the three figures 131. To get the same accuracy by squaring $a$ and $b$ it would be necessary to find $a^2$ to seven significant figures either from a larger table of squares or by ordinary multiplication.

**Factors of $x^2+px+q$.**

Let $x^2+8x+12$ be the product of $(x+a)$ and $(x+b)$. Since

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

the product is $x^2+8x+12$, if

$$a+b=8, \text{ and } ab=12.$$

Therefore to find the two factors we have to find two numbers whose product is 12 and whose sum is 8. To do this we write out a list of the possible factors of 12, namely, $12\times1$, $6\times2$, $4\times3$ and pick out the pair whose sum is 8, in this case $6\times2$.

$$\therefore\ x^2+8x+12=(x+6)(x+2).$$

This work can be set out in the following way :

$$x^2+8x+12= \times \begin{array}{l} x+\cancel{12},\ 6,\ \cancel{4} \\ x+\ \cancel{1},\ 2,\ \cancel{3} \end{array}$$

## Exercise VIII

Factorize :

1. (a) $a^2+ab$. (b) $x^2-xy$. (c) $a^2-h^2$.
2. (a) $2pq-3q^2$. (b) $p^2-q^2$. (c) $4l^2-m^2$.
3. (a) $x^2-9$. (b) $121-r^2$. (c) $121^2-119^2$.

4. (a) $9c^2 - 16r^2$. (b) $8k^2 - 200$. (c) $\frac{1}{4}a^2 - r^2$.

5. (a) $p^2q^2 - a^2b^2$. (b) $p^2q^2 - p^2b^2$. (c) $20p^2q^2 - 45a^2b^2$.

6. (a) $(2x-1)^2 - (x-2)^2$. (b) $(l-x)^2 - (l-x)(l+x)$.

7. (a) $l^4 - 16$. (b) $l^4 - 16l^2$. (c) $(a+2r)^2 - (a-r)^2$.

8. (a) $16x^4 - 81y^4$. (b) $(a+b+c)^2 - (a+b-c)^2$.

9. (a) $\dfrac{t^2}{T^2} - 1$; (b) $r^{2n} - 1$; (c) $k^2 - \dfrac{9}{m^2}$.

Calculate by using factors :

10. (a) $28^2 - 23^2$. (b) $65^2 - 63^2$. (c) $9{\cdot}87^2 - 9{\cdot}81^2$.

11. (a) $152^2 - 148^2$. (b) $289^2 - 286^2$. (c) $7594^2 - 7590^2$.

12. Find the value of $p^2 - q^2$ to 3 sig. fig. when $p = 150{\cdot}0237$ and $q = 150$.

13. Find the value of $\omega^2 - \omega_0^2$ to 3 sig. fig. when $\omega_0 = 2 \times 10^8$ and $\omega - \omega_0 = 12{,}600$.

14. Two concentric circles have radii $r_1$ and $r_2$. If C is the length of the circumference of a circle whose radius is the mean of $r_1$ and $r_2$ and $t$ is the distance between the circles, show that the area between them is C$t$.

Factorize :

15. (a) $ax - 2x - a + 2$. (b) $l^2 + lm + ln + mn$.

16. (a) $ka - kb + lb - la$. (b) $t^2 - 3st - 3s + t$.

17. (a) $(a+b)^2 - c^2$. (b) $a^2 - 2ab + b^2 - c^2$.

18. (a) $a^2 - b^2 + 2bc - c^2$. (b) $4k^2 + 4kl + l^2 - 9m^2$.

19. (a) $x^2 + 5x + 4$. (b) $x^2 - 5x + 4$. (c) $x^2 - 3x - 4$.

20. (a) $x^2 + 23x + 90$. (b) $x^2 - 23x + 90$. (c) $x^2 + 13x - 90$.

21. (a) $p^2 + 16p + 63$. (b) $p^2 - 2p - 63$. (c) $p^2 + 2p - 63$.

22. (a) $y^2 + 8y + 15$. (b) $y^2 + 2y - 15$. (c) $y^2 - 2y - 15$.

23. (a) $x^2 + 3x - 4$. (b) $x^2 + 3x - 28$. (c) $x^2 - 3x - 28$.

24. (a) $l^2 + l - 6$. (b) $l^2 - l - 6$. (c) $l^2 - 7l + 6$.

25. (a) $m^2 - 5m - 14$. (b) $m^2 + 5m - 14$. (c) $m^2 - 15m + 14$.

26. (a) $p^2 + 3pq + 2q^2$. (b) $p^2 - 3pq + 2q^2$. (c) $p^2 - pq - 2q^2$.

27. (a) $l^2 + lm - 2m^2$. (b) $l^2 - lm - 2m^2$. (c) $l^2 - lm - 30m^2$.

28. (a) $p^2q^2 - 5pq + 6$. (b) $p^2 - 5pq + 6q^2$. (c) $p^2q^2 - pq - 6$.

Simplify, using factors :

**29.** $x^2(y-x)+x(y-x)^2$. **30.** $(a-x)^2-(a^2-x^2)$.

**31.** $(l+n)(l^2-x^2)-(l-x)(l^2-n^2)$. **32.** $(r+2t+1)^2-(r-3t+1)^2$.

Factorize :

**33.** (*a*) $1-\cos^2\theta$. (*b*) $\cos^2 A-\sin^2 A$.
(*c*) $(3\cos A+2\sin A)^2-(\cos A+\sin A)^2$.

**34.** (*a*) $4\tan^2\theta-1$. (*b*) $1+\sin\theta+2\cos\theta+2\sin\theta\cos\theta$.
(*c*) $\cos^4 x-\sin^4 x$.

## Factors of $ax^2+bx+c$

$$(3x+2)(2x+1)=3\times 2x^2+(3\times 1+2\times 2)x+2\times 1$$
$$=6x^2+7x+2.$$

Now consider the reverse process of finding the factors of $6x^2+7x+2$.

Let $6x^2+7x+2=(ax+b)(cx+d)$
$$=acx^2+(ad+bc)x+bd.$$

The expression on the right-hand side is the same as the expression on the left if

$$ac=6,\ (ad+bc)=7 \text{ and } bd=2.$$

If we write the letters $a$, $b$, $c$, $d$ in the form $\begin{matrix} a & & b \\ & \times & \\ c & & d \end{matrix}$ then the product of the two left-hand letters has to be 6, the product of the two right-hand letters has to be 2, and the sum of the cross products indicated by the arrow heads has to be 7. Now $6=6\times 1$ and $3\times 2$, and $2=2\times 1$. Therefore, the scheme $\begin{matrix} a & & b \\ & \times & \\ c & & d \end{matrix}$ can be written in the following ways :

$$\begin{matrix} 6 & & 2 \\ & \times & \\ 1 & & 1 \end{matrix}\ ;\quad \begin{matrix} 6 & & 1 \\ & \times & \\ 1 & & 2 \end{matrix}\ ;\quad \begin{matrix} 3 & & 2 \\ & \times & \\ 2 & & 1 \end{matrix}\ ;\quad \begin{matrix} 3 & & 1 \\ & \times & \\ 2 & & 2 \end{matrix}$$

We examine each of these to see in which one the sum of the cross products is 7. In the first it is $6+2=8$, in the second, $12+1=13$, in the third, $3+4=7$. Hence the third arrangement is the correct one, that is, $a=3$, $b=2$, $c=2$ and $d=1$.

$\therefore\ 6x^2+7x+2=(3x+2)(2x+1)$.

The above schemes of factors can be written more compactly

$$\begin{matrix} 6, 3 & \searrow\!\!\nearrow & 2, 1 \\ 1, 2 & \nearrow\!\!\searrow & 1, 2 \end{matrix}$$

When some of the terms are negative we proceed in the same way except that in this case either $b$ or $d$ or both will be negative numbers.

*Example.*—Factorize $5t^2 - 8t - 4$.

Let $5t^2 - 8t - 4 = (at + b)(ct + d)$

$$= act^2 + (ad + bc)t + bd$$

Then $ac = 5$, $bd = -4$ and $ad + bc = -8$.

Therefore, $a$ and $c$ must be 5 and 1, and $b$ and $d$ can be any one of the combinations 4 and $-1$, $-4$ and 1, 1 and $-4$, $-1$ and 4, 2 and $-2$, $-2$ and 2. We write these :

$$\begin{matrix} 5 & \searrow\!\!\nearrow & 4, & -4, & 1, & -1, & 2, & -2. \\ 1 & \nearrow\!\!\searrow & -1, & 1, & -4, & 4, & -2, & 2. \end{matrix}$$

Working out the cross products and adding them we get in turn $-5 + 4 = -1$, $5 - 4 = 1$, $-20 + 1 = -19$, $20 - 1 = 19$, $-10 + 2 = -8$. The last one gives the correct coefficient of $t$. Hence $a = 5$, $c = 1$, $b = 2$, $d = -2$.

$\therefore\ 5t^2 - 8t - 4 = (5t + 2)(t - 2)$.

The factors of $5t^2 - 8th - 4h^2$ are found in the same way. For if $5t^2 - 8th - 4h^2 = (at + bh)(ct + dh)$, $ac = 5$, $bd = -4$, $ad + bc = -8$ as above.

$\therefore\ 5t^2 - 8th - 4h^2 = (5t + 2h)(t - 2h)$.

## Division

We can often determine whether one expression divides exactly into another by factorizing the dividend. Thus

$$\frac{x^2 - x - 12}{x - 4} = \frac{(x - 4)(x + 3)}{x - 4} = x + 3$$

$\therefore\ x - 4$ divides exactly $x + 3$ times into $x^2 - x - 12$.

The division can however be carried out by the method used

in long division in arithmetic, after first arranging both dividend and divisor in descending powers of $x$.

$$\begin{array}{r|l|l} x-4 & x^2-x-12 & x+3 \\ & x^2-4x & \\ \hline & \phantom{x^2-{}}3x-12 & \\ & \phantom{x^2-{}}3x-12 & \\ \hline & \phantom{x^2-3x-1}0 & \end{array}$$

The first step is to divide $x$ into $x^2$; the quotient is $x$ and so we multiply the divisor by $x$, giving $x^2-4x$. Subtracting $x^2-4x$ from the dividend the remainder is $3x-12$. This step shows that

$$\begin{aligned} x^2-x-12 &= x^2-4x+3x-12 \\ &= x(x-4)+3x-12. \end{aligned}$$

Now $x-4$ divides exactly 3 times into $3x-12$.

$$\begin{aligned} \therefore\ x^2-x-12 &= x(x-4)+3(x-4) \\ &= (x+3)(x-4), \end{aligned}$$

or

$$\frac{x^2-x-12}{x-4}=x+3.$$

If the dividend is $x^2-x-10$, the same method gives,

$$\begin{array}{r|l|l} x-4 & x^2-x-10 & x+3. \\ & x^2-4x & \\ \hline & \phantom{x^2-{}}3x-10 & \\ & \phantom{x^2-{}}3x-12 & \\ \hline & \phantom{x^2-3x-1}2 & \end{array}$$

Thus there is a remainder 2. The steps in the division show that:

$$\begin{aligned} x^2-x-10 &= x^2-4x+3x-10 \\ &= x(x-4)+3x-12+2 \\ &= x(x-4)+3(x-4)+12 \\ &= (x+3)(x-4)+2, \end{aligned}$$

or

$$\frac{x^2-x-10}{x-4}=x+3+\frac{2}{x-4}.$$

*Example.*—Divide $4t^3+6t^2-7t+9$ by $2t^2-3t-1$.

$$\begin{array}{r|l|l}
2t^2-3t-1 & 4t^3+6t^2-7t+9 & 2t+6 \\
 & 4t^3-6t^2-2t & \\
\hline
 & \quad 12t^2-5t+9 & \\
 & \quad 12t^2-18t-6 & \\
\cline{2-2}
 & \qquad 13t+15 &
\end{array}$$

We stop at this stage because $2t^2$ will not divide into $13t$ (without introducing negative powers of $t$). Hence the quotient is $2t+6$ and the remainder is $13t+15$. The division shows that:

$$4t^3+6t^2-7t+9=(2t^2-3t-1)(2t+6)+13t+15.$$

**Factors of $a^3-b^3$ and of $a^3+b^3$**

By division

$$\begin{array}{r|l|l}
a-b & a^3-b^3 & a^2+ab+b^2 \\
 & a^3-a^2b & \\
\cline{2-2}
 & \quad a^2b-b^3 & \\
 & \quad a^2b-ab^2 & \\
\cline{2-2}
 & \qquad ab^2-b^3 & \\
 & \qquad ab^2-b^3 & \\
\cline{2-2}
\end{array}$$

$\therefore\ a^3-b^3=(a-b)(a^2+ab+b^2)$.

Similarly, $a^3+b^3=(a+b)(a^2-ab+b^2)$.

*Example.*—Factorize $8x^3-27y^3$.

$$\begin{aligned}
8x^3-27y^3&=(2x)^3-(3y)^3\\
&=(2x-3y)\{(2x)^2+2x\times 3y+(3y)^2\}\\
&=(2x-3y)(4x^2+6xy+9y^2).
\end{aligned}$$

## Exercise IX

Factorize:

| | | |
|---|---|---|
| 1. $2x^2+3x+1$. | 2. $3t^2+7t+4$. | 3. $3t^2+8t+4$. |
| 4. $4y^2-7y-2$. | 5. $2m^2-3mn-2n^2$. | 6. $4x^2-15x-4$. |
| 7. $6z^2-z-2$. | 8. $3l^2-5l-2$. | 9. $3+8k+5k^2$. |

Divide by factorizing and by long division :

**10.** $x^2-6x-7$ by $x+1$. **11.** $3y^2+5y+2$ by $3y+2$.

**12.** $a^2-7ab-8b^2$ by $a-8b$. **13.** $8p^2-10pq-3q^2$ by $2p-3q$.

Find the quotient and remainder when :

**14.** $x^3-2x-1$ is divided by $x-1$.

**15.** $6x^3-7x^2+5x-7$ is divided by $2x-3$.

**16.** $4y^3+y^2+2y-2$ is divided by $y^2-2$.

**17.** $t^3-2t+5$ is divided by $3t-1$.

**18.** Prove $\dfrac{x^2+4x+5}{x+1}=x+3+\dfrac{2}{x+1}$.

**19.** Prove $\dfrac{y^2-7}{y+1}=y-1-\dfrac{6}{y+1}$.

Factorize :

**20.** $y^3-1$. **21.** $x^3-a^3$. **22.** $27+p^3$. **23.** $20l^3+160$.

**24.** Divide $x-l$ into $x^3-2lx^2+l^3$. The deflection at a point of a beam is proportional to $x^4-2lx^3+l^3x$. Use your first result to factorize this expression.

**25.** Prove that $(\alpha^2+\beta^2)^2=(\alpha^2-\beta^2)^2+4\alpha^2\beta^2$.

If $\alpha^2-\beta^2=GR-LC\omega^2$ and $2\alpha\beta=(GL+RC)\omega$ prove that

$$\alpha^2+\beta^2=\sqrt{(R^2+L^2\omega^2)(G^2+C^2\omega^2)}.$$

**26.** Assuming that the volume of a sphere of radius $r$ is $\frac{4}{3}\pi r^3$, prove that the volume between two concentric spheres of radii R and $r$ is $\frac{4}{3}\pi(R-r)(R^2+Rr+r^2)$ and show that, when R is nearly equal to $r$, this volume is nearly $4\pi r^2(R-r)$.

A spherical balloon has a thickness of 0·005 in. when its radius is 1 ft. Calculate the volume of the material of which it is made. How thick will it be when its radius is 5 ft. ?

## Simplification of fractions

A fraction can often be simplified by finding the factors of the denominator and numerator and then dividing both the denominator and numerator by every factor common to both.

*Example.*—Simplify (i) $\dfrac{a^2bc}{abc^2}$; (ii) $\dfrac{2k^2-k}{k^2+3k}$; (iii) $\dfrac{x^2y+2xy}{x^2-4}$.

(i) $$\frac{a^2bc}{abc^2}=\frac{a\cancel{(abc)}}{c\cancel{(abc)}}=\frac{a}{c}.$$

(ii) $$\frac{2k^2-k}{k^2+3k}=\frac{\cancel{k}(2k-1)}{\cancel{k}(k+3)}=\frac{2k-1}{k+3}.$$

(iii) $$\frac{x^2y+2xy}{x^2-4}=\frac{xy\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}=\frac{xy}{x-2}.$$

*Example.*—Simplify (i) $\dfrac{a^2-r^2}{a-2r}\div\dfrac{a+r}{a^2-2ar}$; (ii) $\dfrac{x^2-2x-3}{2x^2-x-3}$.

(i) $$\frac{a^2-r^2}{a-2r}\div\frac{a+r}{a^2-2ar}=\frac{a^2-r^2}{a-2r}\times\frac{a^2-2ar}{a+r}.$$

$$=\frac{\cancel{(a+r)}(a-r)a\cancel{(a-2r)}}{\cancel{(a-2r)}\cancel{(a+r)}}$$

$$=(a-r)a$$

$$=a^2-ar$$

(ii) $x^2-2x-3=(x-3)(x+1)$.

$2x^2-x-3=(2x-3)(x+1)$. $\left[\begin{matrix}2 & \times & -3, & 3, & -1, & 1.\\ 1 & & 1, & -1, & 3, & -3.\end{matrix}\right]$

$$\therefore \frac{x^2-2x-3}{2x^2-x-3}=\frac{(x-3)(x+1)}{(2x-3)(x+1)}=\frac{x-3}{2x-3}.$$

## L.C.M. of more than two expressions

*Example.*—Find the L.C.M. of $x^2-x-2$, $2x^2-x-6$, $2x^2+5x+3$.

$$x^2-x-2=(x-2)(x+1)$$
$$2x^2-x-6=(2x+3)(x-2)$$
$$2x^2+5x+3=(2x+3)(x+1)$$

Hence the lowest common multiple into which each of the three expressions will divide is :

$$(x-2)(x+1)(2x+3).$$

## Addition and subtraction of fractions

We have seen on p. 8 that fractions are added or subtracted by bringing them to a common denominator, which is the L.C.M. of their denominators. Some harder examples are given below.

*Example.*—Express $\dfrac{3}{x+1}-\dfrac{6}{x+2}$ as a single fraction.

L.C.M. of denominators is $(x+1)(x+2)$.

$$\therefore \frac{3}{x+1}-\frac{6}{x+2}=\frac{3(x+2)}{(x+1)(x+2)}-\frac{6(x+1)}{(x+1)(x+2)}$$

$$=\frac{3x+6-6x-6}{(x+1)(x+2)}$$

$$=-\frac{3x}{(x+1)(x+2)}.$$

*Example.*—Simplify $\dfrac{t+3}{t^2-t}-\dfrac{2t}{2t^2-3t+1}$.

$$t^2-t=t(t-1),$$
$$2t^2-3t+1=(2t-1)(t-1).$$

$\therefore$ L.C.M. of denominators $=t(t-1)(2t-1)$.

$$\therefore \text{expression}=\frac{(2t-1)(t+3)}{t(t-1)(2t-1)}-\frac{2t^2}{t(t-1)(2t-1)}$$

$$=\frac{2t^2+5t-3-2t^2}{t(t-1)(2t-1)}$$

$$=\frac{5t-3}{t(t-1)(2t-1)}.$$

*Example.*—Simplify $\dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{b^2}-\dfrac{1}{a^2}}\times\dfrac{\dfrac{1}{ab}+\dfrac{1}{b^2}}{2a}$

$$\text{Expression}=\frac{\dfrac{a-b}{ab}}{\dfrac{a^2-b^2}{a^2b^2}}\times\frac{\dfrac{b+a}{ab^2}}{2a}$$

$$=\frac{(a-b)a^2\cancel{b^2}}{ab(a^2-b^2)}\times\frac{(a+b)}{2\cancel{a^2b^2}}$$

$$=\frac{(a-b)(a+b)}{2ab(a^2-b^2)}$$

$$=\frac{1}{2ab}, \text{ since } a^2-b^2=(a+b)(a-b).$$

In an example of this type it is important to note the difference between the horizontal lines in the fraction $\dfrac{\frac{b+a}{ab^2}}{2a}$. This expression with the bottom line longer than the upper line means $\dfrac{b+a}{ab^2}\div 2a$, i.e. $\dfrac{b+a}{2a^2b^2}$, but the expression $\dfrac{b+a}{\frac{ab^2}{2a}}$ means $(b+a)\div\dfrac{ab^2}{2a}$, which is $\dfrac{2a(b+a)}{ab^2}$.

The same method is used in simplifying expressions containing fractional indices, but each term should be expressed without negative indices before carrying out the addition or subtraction.

*Example.*—Simplify $(r^2+a^2)^{-\frac{3}{2}}-3r^2(r^2+a^2)^{-\frac{5}{2}}$.

$$\text{Expression}=\frac{1}{(r^2+a^2)^{\frac{3}{2}}}-\frac{3r^2}{(r^2+a^2)^{\frac{5}{2}}}.$$

Since $(r^2+a^2)^{\frac{5}{2}}=(r^2+a^2)^{\frac{3}{2}}\times(r^2+a^2)$, it is the L.C.M. of the denominators.

$$\therefore \text{ expression}=\frac{r^2+a^2}{(r^2+a^2)^{\frac{5}{2}}}-\frac{3r^2}{(r^2+a^2)^{\frac{5}{2}}}$$

$$=\frac{a^2-2r^2}{(r^2+a^2)^{\frac{5}{2}}}.$$

**Equations involving fractions**

Such equations are simplified by multiplying each term by the L.C.M. of the denominators.

*Example.*—Solve the equation $\dfrac{4}{x(x+3)} = \dfrac{1}{x^2+2x-3}$.

$$x^2+2x-3=(x+3)(x-1).$$

$\therefore$ L.C.M. of denominators $= x(x+3)(x-1)$.

Multiplying each term of the equation by this L.C.M.,

$$\frac{4\cancel{x(x+3)}(x-1)}{\cancel{x(x+3)}} = \frac{x\cancel{(x+3)(x-1)}}{\cancel{(x+3)(x-1)}}$$

$$\therefore\ 4(x-1)=x$$

$$\therefore\ 4x-4=x$$

$$\therefore\ 3x=4$$

$$\therefore\ x=\tfrac{4}{3}$$

*Example.*—Express $r$ as the subject of the formula:

$$\frac{1}{x+r}+\frac{1}{y+r}=\frac{2}{x+y+r}.$$

Multiplying by the L.C.M., viz. $(x+r)(y+r)(x+y+r)$,

$$(y+r)(x+y+r)+(x+r)(x+y+r)=2(x+r)(y+r)$$

$$\therefore\ \cancel{xy}+y^2+yr+xr+yr+\cancel{r^2}+x^2+\cancel{xy}+xr+xr+yr+\cancel{r^2}$$
$$=\cancel{2xy}+2xr+2yr+\cancel{2r^2}$$

$$\therefore\ 3yr+3xr+x^2+y^2=2xr+2yr$$

$$\therefore\ xr+yr=-(x^2+y^2)$$

$$\therefore\ r(x+y)=-(x^2+y^2)$$

$$\therefore\ r=-\frac{x^2+y^2}{x+y}.$$

## Exercise X

Simplify:

1. $\dfrac{ab^2x}{ax^2}$.
2. $\dfrac{16lmn^2}{12l^2m^2}$.
3. $\dfrac{\frac{8}{15}\pi r^5}{\frac{4}{3}\pi r^3}$.
4. $\dfrac{(6xr)^3}{(3xr^2)^2}$.
5. $\dfrac{x^2-xy}{x^2+xz}$.
6. $\dfrac{p^2-4}{4p-8}$.

7. $\dfrac{a^2-ab}{ac-bc}$.

8. $\dfrac{x^2-9}{x^2-2x-3}$.

9. $\dfrac{r^2-s^2}{(r+s)^2}$.

10. $\dfrac{p-q}{p^2+pq-2q^2}$.

11. $\dfrac{z^2-2z-3}{z^2+z-12}$.

12. $\dfrac{x^2-4x}{x^2+x-20}$.

13. $\dfrac{y^2-4}{y+3}\times\dfrac{y^2-9}{y-2}$.

14. $\dfrac{(ac-2c^2)^2}{a^2-ac-2c^2}\times\dfrac{a+c}{a^2-4c^2}$.

15. $\dfrac{x+3}{x+2}\div\dfrac{x+1}{x+2}$.

16. $\dfrac{r^2-x^2}{r^2x}\div\dfrac{rx+x^2}{r^3x^2}$.

17. $\dfrac{\frac{1}{p}-\frac{1}{q}}{\frac{1}{p^2}-\frac{1}{q^2}}$.

18. $\left(\dfrac{2}{a}-\dfrac{3}{b}\right)\div\left(\dfrac{a}{2}-\dfrac{b}{3}\right)$.

19. $\dfrac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{r}+s}\times\dfrac{r^2s^2-1}{r^2-s^2}$.

20. $\dfrac{(m-n)^2+4mn}{(m+1)^2-(n-1)^2}$.

21. $\dfrac{x-3}{2x^2-7x+3}$.

22. $\dfrac{3y^2+4y}{3y^2+y-4}$.

23. $\dfrac{m^2-2mn+n^2}{2m^2+mn-3n^2}$.

24. $\dfrac{2l^2-3l+1}{l^2-3l+2}$.

Express as single fractions :

25. $\dfrac{1}{x+3}-\dfrac{1}{x+4}$.

26. $\dfrac{1}{t}+\dfrac{2}{3-t}$.

27. $\dfrac{4}{3ab}-\dfrac{5}{6bc}$.

28. $\dfrac{m^2+2}{m^2+m}-\dfrac{m-2}{m}$.

29. $\dfrac{1}{l-1}-\dfrac{l+1}{l^2-l}$.

30. $\dfrac{3-y}{1-3y}-\dfrac{3+y}{1+3y}$.

31. $\dfrac{2a}{a+b}+\dfrac{2b}{a-b}+\dfrac{a^2-2b^2}{a^2-b^2}$.

32. $\dfrac{1}{y}-\dfrac{2}{y^2+2y}$.

33. $\dfrac{1}{(r-s)^2}-\dfrac{1}{(r+s)^2}$.

34. $\dfrac{5x+4}{x^2-2x}-\dfrac{7}{x-2}$.

35. $\dfrac{z}{2z^2-z-1}-\dfrac{1}{2z+1}$.

36. $\dfrac{p+q}{p-2q}-\dfrac{p-q}{p+2q}$.

37. $\dfrac{1}{(x-1)(x-2)}-\dfrac{1}{(x-2)(x+3)}$. 38. $\dfrac{2}{x^2-x}-\dfrac{4}{2x^2-3x+1}$.

39. $\dfrac{1}{t^2-2t-15}-\dfrac{1}{3t^2+10t+3}$. 40. $\dfrac{x^3+y^3}{x^4-y^4}-\dfrac{1}{x-y}$.

Simplify the following, if possible :

41. (*a*) $\dfrac{x^2+xy}{x+y}$. (*b*) $\dfrac{x^2+xy}{x^2+y^2}$.

42. (*a*) $\dfrac{a+2b}{a+b}$. (*b*) $\dfrac{2a+2b}{a+b}$.

43. (*a*) $\dfrac{m^2+n^2}{m+n}$. (*b*) $\dfrac{m^2+2mn+n^2}{m+n}$.

44. (*a*) $\dfrac{6mn}{3m^2n^2}$. (*b*) $\dfrac{6m+n}{3m^2+n^2}$.

45. (*a*) $\dfrac{k^2-l^2}{(k+l)^2}$. (*b*) $\dfrac{k^2-l^2}{k^2+l^2}$.

46. Using $\sin^2 A+\cos^2 A=1$, prove :

$$\frac{(1+\sin A)^2}{\cos^2 A}=\frac{1+\sin A}{1-\sin A}.$$

Prove that :

47. $t^{-\frac{3}{2}}-6t^{-\frac{5}{2}}+9t^{-\frac{7}{2}}=(t-3)^2/t^{\frac{7}{2}}$.

48. $4(x+2)^{\frac{1}{2}}-\frac{4}{3}(x+2)^{\frac{3}{2}}+\frac{1}{5}(x+2)^{\frac{5}{2}}=\frac{1}{15}(3x^2-8x+32)\sqrt{x+2}$.

49. $(a-b)^{\frac{3}{2}}+2b(a-b)^{\frac{1}{2}}+b^2(a-b)^{-\frac{1}{2}}=a^2/\sqrt{a-b}$.

Solve the equations :

50. $\dfrac{2}{x(x+3)}=\dfrac{1}{x(x+1)}$. 51. $\dfrac{x+1}{(x-1)(x-2)}-\dfrac{x+2}{(x-2)(x+3)}=0$.

52. $\dfrac{3}{t+1}-\dfrac{1}{t(t+1)}=\dfrac{1}{t}$. 53. $\dfrac{p+2}{p-2}-\dfrac{p-2}{p+2}=\dfrac{16}{p^2-4}$.

54. Simplify $\frac{1}{2}b\cdot\dfrac{Wab}{a+b}\cdot\frac{2}{3}b\,.+\frac{1}{2}a\cdot\dfrac{Wab}{a+b}\cdot(b+\frac{1}{3}a)$.

55. Two expressions for the same bending moment are :

$$\frac{wx^2}{2}-\frac{2wl(l-b)(x-a)}{2l-a-b} \text{ and } \frac{w(2l-x)^2}{2}-\frac{2wl(l-a)(2l-x-b)}{2l-a-b}.$$

Prove that these expressions are equal.

**56.** A frustum of a cone has end radii $r_1$ and $r_2$ and length $h$. Its volume V is given by $V=\frac{1}{3}\pi(r_2{}^2h_2-r_1{}^2h_1)$, where $h_1/r_1=h_2/r_2=h/(r_2-r_1)$. Prove $V=\frac{1}{3}\pi(r_2{}^2+r_2r_1+r_1{}^2)h$. Is V equal to the volume of a cylinder of length $h$ and radius equal to the mean of $r_1$ and $r_2$?

**57.** The radius of gyration $k$ of a hollow cylinder of internal radius $a$ and external radius $b$ is given by $k^2=\dfrac{\frac{1}{2}\pi(b^4-a^4)}{\pi(b^2-a^2)}$. Simplify this expression. Show that, if $(b-a)$ is small, $k \simeq a$.

**58.** If a resistance $r$ is placed in parallel with resistances $r_1$ and $r_2$, which are in series, the combined resistance R is given by $\dfrac{1}{R}=\dfrac{1}{r}+\dfrac{1}{r_1+r_2}$, but, if $r$ is in series with $r_1$ and $r_2$ which are in parallel, $R=r+\dfrac{1}{\dfrac{1}{r_1}+\dfrac{1}{r_2}}$. Find R from each of these equations as a single fraction.

# CHAPTER III

## SOLUTION OF QUADRATIC AND OTHER EQUATIONS

### Quadratic equations

If
$$x^2-3x+2=0,$$
$$(x-1)(x-2)=0$$
$$\therefore x-1=0 \quad \text{or} \quad x-2=0$$
$$\therefore x=1 \quad \text{or} \quad x=2$$
$$\therefore x=1 \quad \text{or} \quad 2.$$

An equation like $x^2-3x+2=0$, which is true for two values of the unknown letter $x$, that is, has two roots, is called a "quadratic equation." The simplest type of quadratic equation is one like $4x^2-9=0$, which has no term containing the first power of the unknown $x$. It can be solved in either of the following ways.

(i) Factorize $4x^2-9$, then

$$(2x+3)(2x-3)=0$$
$$\therefore\ 2x+3=0 \quad \text{or } 2x-3=0$$
$$\therefore\ 2x=-3, \quad \text{or} \quad 2x=3$$
$$\therefore\ x=-\tfrac{3}{2} \quad \text{or} \quad x=\tfrac{3}{2}$$
$$\therefore\ x=-\tfrac{3}{2} \quad \text{or} \quad \tfrac{3}{2}.$$

(ii) Add 9 to both sides of the equation, then

$$4x^2-9+9=9$$
$$\therefore\ 4x^2=9$$
$$\therefore\ (2x)^2=3^2 \text{ or } (-3)^2.$$

Now take the square root of each side,

$$2x=3 \quad \text{or} \quad 2x=-3$$
$$\therefore\ x=\tfrac{3}{2} \quad \text{or} \quad -\tfrac{3}{2}.$$

$\pm 3$ is written for " $+3$ or $-3$ " and the two lines above are written :

$$2x=\pm 3 \qquad \therefore\ x=\pm\tfrac{3}{2}.$$

### Solution of a quadratic equation by factors

Take all the terms of the equation to the left-hand side; then, if the left-hand side has easy factors, the roots of the equation are found by equating each factor to zero, as in (i) above.

*Example.*—Solve the equation $3x^2=2x+5$.

$$3x^2-2x-5=0$$
$$\therefore\ (3x-5)(x+1)=0$$
$$\therefore\ 3x-5=0 \quad \text{or} \quad x+1=0$$
$$\therefore\ x=\tfrac{5}{3} \quad \text{or} \quad -1.$$

*Example.*—Find $t$ if $t^2-at+2bt-2ab=0$.

$$t(t-a)+2b(t-a)=0$$
$$\therefore (t-a)(t+2b)=0$$
$$\therefore\ t-a=0 \quad \text{or} \quad t+2b=0$$
$$\therefore\ t=a \quad \text{or} \quad -2b.$$

**Solution of a quadratic equation by the "method of completing the square"**

Since

$$(x+\tfrac{1}{2}p)^2 = x^2 \quad + \quad 2x \times \tfrac{1}{2}p \quad + \quad (\tfrac{1}{2}p)^2 = x^2 + px + (\tfrac{1}{2}p)^2,$$

$x^2 + px$ is converted into a perfect square by adding $(\frac{1}{2}p)^2$. This is shown geometrically in Fig. 13, in which

$$x^2 + px = \text{area of square ABCD} + \text{area of rectangle BEFC} \\ + \text{ area of rectangle CHKD}.$$

The addition of $(\frac{1}{2}p)^2$ to this area means the addition of the square CFGH, which makes up the total area of the square AEGK, which has an area $(x+\frac{1}{2}p)^2$.

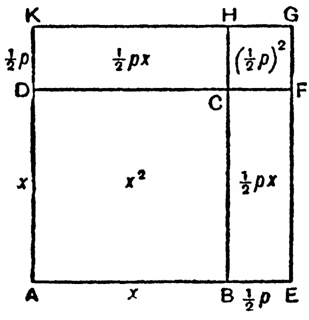



FIG. 13.

*Example.*—What must be added to $x^2 + 3x$ to make the new expression a perfect square?

Here $p = 3$ and so $\left(\dfrac{p}{2}\right)^2 = (\frac{3}{2})^2 = \frac{9}{4}$. This makes the new expression

$$x^2 + 3x + (\tfrac{3}{2})^2 = (x + \tfrac{3}{2})^2.$$

*Example.*—Solve the equation $x^2 + 3x - 5 = 0$.

$$x^2 + 3x = 5$$

Adding $(\frac{3}{2})^2$ to both sides,

$$x^2+3x+(\tfrac{3}{2})^2=5+(\tfrac{3}{2})^2=5+\tfrac{9}{4}$$

$$\therefore\ (x+\tfrac{3}{2})^2=\tfrac{29}{4}$$

$$\therefore\ x+\tfrac{3}{2}=\pm\sqrt{\frac{29}{4}}=\pm\frac{\sqrt{29}}{2}\simeq\pm\frac{5{\cdot}385}{2}$$

$$\therefore\ x\simeq\frac{5{\cdot}385-3}{2}\simeq1{\cdot}192\ \text{ or }\ \frac{-5{\cdot}385-3}{2}\simeq-4{\cdot}192.$$

This example shows that to solve the equation $x^2+px+q=0$:

I. Write the equation in the form $x^2+px=-q$.

II. Add to both sides $(\frac{1}{2}p)^2$, that is the "square of half the coefficient of $x$." This makes the left-hand side the square of $(x+\frac{1}{2}p)$.

The above method of solution is called the "method of completing the square." It can be used in all cases whatever the values of $p$ and $q$, whether positive or negative.

*Example.*—Solve the equation $y^2+4{\cdot}12=7{\cdot}66y$.

$$y^2-7{\cdot}66y=-4{\cdot}12.$$

Add $\left(\frac{7{\cdot}66}{2}\right)^2=3{\cdot}83^2$ to both sides,

$$y^2-7{\cdot}66y+\left(\frac{7{\cdot}66}{2}\right)^2=3{\cdot}83^2-4{\cdot}12=14{\cdot}67-4{\cdot}12$$

$$\therefore\ (y-3{\cdot}83)^2=10{\cdot}55=3{\cdot}25^2$$

$$\therefore\ y-3{\cdot}83=\pm3{\cdot}25$$

$$\therefore\ y=3{\cdot}83-3{\cdot}25=0{\cdot}58$$

$$\text{or}\quad y=3{\cdot}83+3{\cdot}25=7{\cdot}08.$$

If $x^2$ has a coefficient we divide the equation by it so as to make the coefficient of $x^2$ unity before completing the square.

*Example.*—Solve the equation $3x^2+x-7=0$.

Dividing by 3,

$$x^2+\tfrac{1}{3}x-\tfrac{7}{3}=0.$$

$$\therefore\ x^2+\tfrac{1}{3}x+(\tfrac{1}{6})^2=\tfrac{7}{3}+(\tfrac{1}{6})^2=\tfrac{7}{3}+\tfrac{1}{36}$$

$$\therefore\ (x+\tfrac{1}{6})^2=\frac{84+1}{36}=\frac{85}{36}$$

$$\therefore\ x+\tfrac{1}{6}=\pm\sqrt{\frac{85}{36}}=\frac{\pm\sqrt{85}}{6}\simeq\pm\frac{9\cdot22}{6}$$

$$\therefore\ x\simeq\frac{9\cdot22-1}{6}=\frac{8\cdot22}{6}=1\cdot37$$

$$\text{or}\qquad x\simeq\frac{-9\cdot22-1}{6}=\frac{-10\cdot22}{6}\simeq-1\cdot70.$$

*Example.*—Make $p$ the subject of the formula :

$$Lp^2+Rp+\frac{1}{C}=0.$$

$$p^2+\frac{R}{L}p+\frac{1}{LC}=0$$

$$\therefore\ p^2+\frac{R}{L}p=-\frac{1}{LC}$$

$$\therefore\ p^2+\frac{R}{L}p+\left(\frac{R}{2L}\right)^2=-\frac{1}{LC}+\left(\frac{R}{2L}\right)^2=-\frac{1}{LC}+\frac{R^2}{4L^2}$$

$$\therefore\ \left(p+\frac{R}{2L}\right)^2=\frac{-4L+R^2C}{4L^2C}$$

$$\therefore\ p+\frac{R}{2L}=\pm\sqrt{\frac{R^2C-4L}{4L^2C}}$$

$$=\pm\frac{1}{2L}\sqrt{\frac{R^2C-4L}{C}}$$

$$=\pm\frac{1}{2L}\sqrt{R^2-\frac{4L}{C}}$$

$$\therefore\ p=\frac{-R\pm\sqrt{R^2-\frac{4L}{C}}}{2L}.$$

**Solution of $ax^2+bx+c=0$**

$ax^2+bx+c=0$ *is the typical quadratic equation.* To sum up, the methods of solving it are :

either I. Factorize $ax^2+bx+c$ and equate each factor to zero;

or II. Divide by $a$ ; take the term without $x$ to the right-hand side ; add ($\frac{1}{2}$ coefficient of $x)^2$ to both sides ; take the square root of each side.

By using the second method we get :

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

$$\therefore\ x^2+\frac{b}{a}x=-\frac{c}{a}$$

$$\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

$$\therefore\ x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This may be remembered as a formula for the roots, but the student should not make use of it in this way until he has mastered the method of completing the square.

*Example.*—Solve, by rule, $3x^2-2x-6=0$.

Here $a=3$, $b=-2$, $c=-6$

$$\therefore x=\frac{-(-2)\pm\sqrt{(-2)^2-4.3(-6)}}{2.3}$$

$$=\frac{2\pm\sqrt{4+72}}{6}$$

$$=\frac{2\pm\sqrt{76}}{6}$$

$$=\frac{2\pm 8{\cdot}718}{6}$$

$$\therefore\ x=\frac{10{\cdot}718}{6}\simeq 1{\cdot}786$$

$$\text{or}\quad x=-\simeq\frac{6{\cdot}718}{6}\simeq -1{\cdot}120.$$

### Factors of $ax^2+bx+c$

Suppose $3x^2-2x-6=3(x-a)(x-b)$, one factor being 3 so as to make the coefficient of $x^2$ equal to 3. Then $3x^2-2x-6=0$ when $x=a$ or $b$. Hence $a$ and $b$ must be the roots found above, namely $\dfrac{1+\sqrt{19}}{3}$ and $\dfrac{1-\sqrt{19}}{3}$.

$$\therefore\ 3x^2-2x-6=3\left(x-\frac{1+\sqrt{19}}{3}\right)\left(x-\frac{1-\sqrt{19}}{3}\right)$$

$$\text{or}\simeq 3(x-1{\cdot}786)(x+1{\cdot}120).$$

### Graphical solution of a quadratic equation

A quadratic equation may also be solved graphically; but since the roots can be found much more accurately by algebra the graphical method should not generally be used. However, it is very good practice in drawing graphs to solve a quadratic equation by means of a graph and compare the answers with those found by calculation; moreover, the experience so gained is helpful when using graphs to solve harder equations which cannot be solved by using a formula. As an example we will solve graphically the equation $3x^2-2x-6=0$, which has been solved by calculation in the example above.

Write $y=3x^2-2x-6$ and make a table of values of $y$ for a range of values of $x$, say from $-3$ to 3.

| $x$ | .. | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|---|
| $3x^2$ | .. | 27 | 12 | 3 | 0 | 3 | 12 | 27 |
| $-2x$ | .. | 6 | 4 | 2 | 0 | $-2$ | $-4$ | $-6$ |
| $-6$ | .. | $-6$ | $-6$ | $-6$ | $-6$ | $-6$ | $-6$ | $-6$ |
| $y$ | .. | 27 | 10 | $-1$ | $-6$ | $-5$ | 2 | 15 |

GRAPH OF $y=3x^2-2x-6$

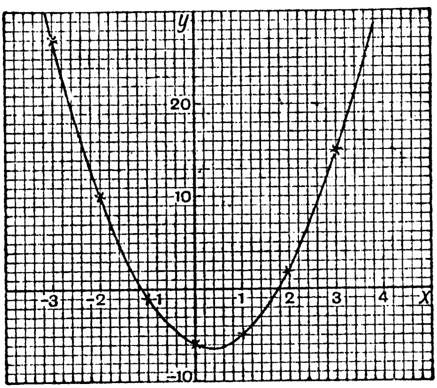



Fig. 14.

From this table it is clear that $y=0$ between $x=-2$ and $-1$ and again between 1 and 2, so that the range chosen includes both roots of the equation.

The graph shows that the two values of $x$ which make $y=0$ are approximately $-1{\cdot}15$ and $1{\cdot}75$.

If we wish to find either root more accurately, we have to tabulate the values of $y$ for a set of values $x$ near the root, and draw a graph from this new table to a much larger scale. In the present example it is certain that the positive root lies

between 1·7 and 1·8, so we tabulate the values of $y$ for $x = 1{\cdot}7$, 1·75, 1·8 say.

| $x$ | .. | 1·7 | 1·75 | 1·8 |
|---|---|---|---|---|
| $3x^2$ | .. | 8·670 | 9·189 | 9·720 |
| $-2x$ | .. | $-3{\cdot}4$ | $-3{\cdot}5$ | $-3{\cdot}6$ |
| $-6$ | .. | $-6$ | $-6$ | $-6$ |
| Sum $= y$ | .. | $-0{\cdot}730$ | $-0{\cdot}311$ | 0·120 |

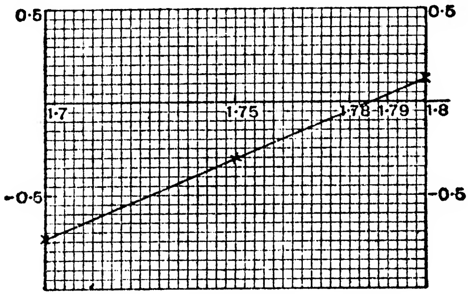



Fig. 15.

The graph drawn from this table is shown in Fig. 15, and it shows that $y = 0$ at 1·786 nearly, which agrees with the

TYPICAL PARABOLAS

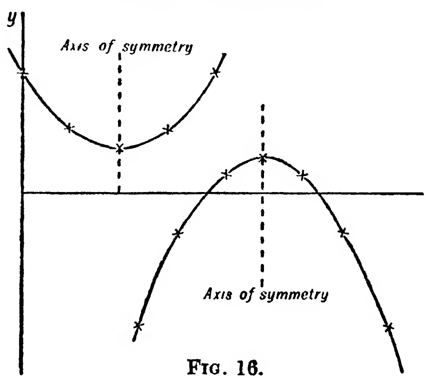



Fig. 16.

value 1·786 found by the formula. This graph is almost a straight line; and so only a very small error is made in taking it to be a straight line.

Whatever the values of $a$, $b$ and $c$, the graph of $y = ax^2 + bx + c$ is of a similar shape to the graph in Fig. 14, but it is the other way up when $a$ is negative. It is symmetrical about the vertical line through the lowest, or highest point. Two typical graphs are shown in Fig. 16. Such curves are called "parabolas."

## Equations of some other curves

*The circle.*

Fig. 17 shows a point $(x, y)$ at a distance $a$ from the point $(0, 0)$. If the scales on the axes of $x$ and $y$ are the same, it follows by Pythagoras' theorem that $x^2 + y^2 = a^2$. This equation is satisfied wherever $(x, y)$ is on the circle. Hence the circle is the graph of the equation. We call the equation "the equation of the circle." Since $y^2 = a^2 - x^2$, $y = \pm\sqrt{a^2 - x^2}$. This shows algebraically what is obvious from the figure, namely, that for any value of $x$ there are two points on the circle given by equal and opposite values of $y$.

THE CIRCLE $x^2+y^2=a^2$

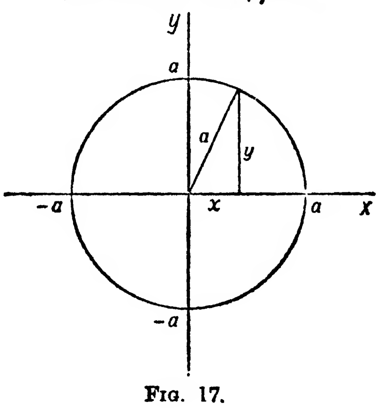



FIG. 17.

*The ellipse.*

Let the ordinate at every point on the circle above be reduced in the ratio $b/a$. Then the locus of the points formed in this way is called an ellipse. If Y is the ordinate of a point P on the circle, and $y$ the ordinate of the point Q found from P by reducing Y in the ratio $b/a$, $y=bY/a$ or $Y=ay/b$. But, since $(x, Y)$ is on the circle, $x^2+Y^2=a^2$:

$$\therefore\ x^2+\frac{a^2y^2}{b^2}=a^2$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

This is therefore the equation of the ellipse.

THE ELLIPSE $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

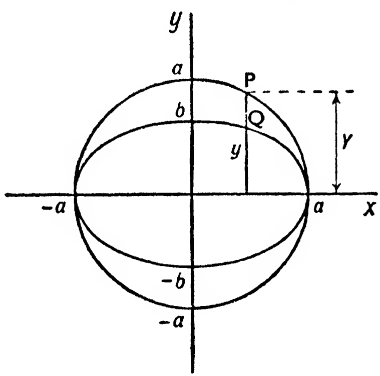



FIG. 18.

*The hyperbola.*

The graph of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is as shown in Fig. 19. It is called a hyperbola.

THE HYPERBOLA $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

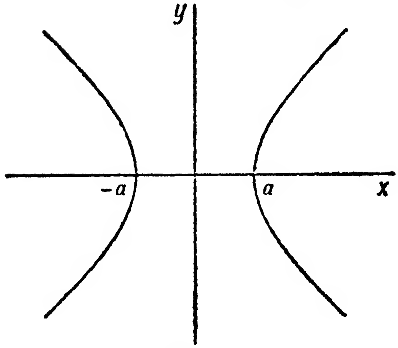



FIG. 19.

*Conic sections.*

The circle, ellipse, hyperbola and parabola are all sections of a double cone by different planes. Fig. 20 shows elliptic, hyperbolic and parabolic sections.

SECTIONS OF A CONE

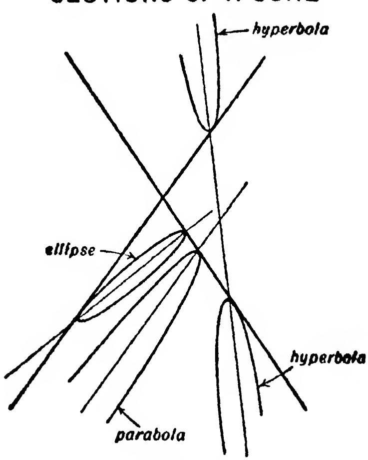



FIG. 20.

## Exercise XI

Solve the equations :

1. $2x^2=3$. 2. $4=\dfrac{1}{9y^2}$. 3. $\dfrac{r}{4}=\dfrac{5}{r}$. 4. $\dfrac{4}{(t-1)^2}=9$.

Solve, by using factors :

5. $x^2-3x+2=0$.
6. $y^2-6y+5=0$.
7. $2r^2-r-3=0$.
8. $12m^2+29m-8=0$.
9. $4x^2+8x+3=0$.
10. $5(l^2-1)=2l(1-l)$.
11. $t^2=t+6$.
12. $x(x+9)=22$.
13. $\dfrac{1}{z+1}-\dfrac{4}{z}=\dfrac{1}{z+3}$.
14. $\dfrac{k+6}{k-4}=\dfrac{1}{k}$.

Solve by using factors and by completing the square :

15. $x^2-4x-5=0$.
16. $n^2+6n+8=0$.
17. $y^2-5y-24=0$.
18. $r^2+9r+8=0$.

19. If $x^2+6x+a$ is a perfect square, what must the value of $a$ be ?

20. If $x^2-\frac{2}{3}x+a$ is a perfect square what must the value of $a$ be ?

21–30. Solve questions 5–14 by completing the square.

Solve by completing the square, and also by using a formula :

31. $x^2+x=4$.
32. $2x^2+5x+1=0$.
33. $3t^2+4t-9=0$.
34. $4v^2=v+1$.
35. $(k+1)^2+k^2=10k$.
36. $1{\cdot}6x^2-6{\cdot}2x-2{\cdot}7=0$.
37. $\sqrt{2}y^2-(1+2\sqrt{2})y-1=0$.
38. $0{\cdot}84t^2+2{\cdot}15t+0{\cdot}18=0$.

39. If $x^2-2ax=b^2-a^2$ prove that $x=a+b$ or $a-b$.

40. Solve the equation $lh=h^2+k^2$ for $h$ and show that the sum of the roots is $l$.

41. If $p^2-(p_x+p_y)p-q^2+p_x p_y=0$, prove that

$$p=\tfrac{1}{2}\{(p_x+p_y)\pm\sqrt{(p_x-p_y)^2+4q^2}\}.$$

Solve the following equations graphically and by calculation and compare your answers :

42. $x^2-6x+3=0$.
43. $2t^2+5t-1=0$.
44. $m^2-5m-7=0$.
45. $r^2+20r-50=0$.

46. Solve the equation $cx^2+ax-b=0$.

47. Find the values of $x$ which make $\dfrac{1}{x}+\dfrac{1}{c-x}=\dfrac{1}{f}$.

48. Make (i) $r$, (ii) $t$, the subject of the formula:

$$W=\pi sl\{(r+t)^2-r^2\}.$$

49. If $4x^2+9y^2=36$, show that $y=\pm\frac{2}{3}\sqrt{9-x^2}$. Make a table of values of $y$ from $x=-3$ to 3, and hence draw the graph of $y$ against $x$. What is the name of this curve?

50. Draw the graphs of the circle $x^2+y^2=9$ and the ellipse $x^2+4y^2=9$ using the same axes for both graphs. Prove that the ellipse is obtained from the circle by halving each ordinate.

Draw the graphs of the following equations and state the name of each curve.

51. $y=4x(3-x)$. 52. $\frac{1}{4}x^2-\frac{1}{2}y^2=1$. 53. $\frac{1}{4}x^2+\frac{1}{2}y^2=1$.

54. $y^2=3x$. 55. $y=1-6x+4x^2$. 56. $y^2-3x^2=4$.

57. Draw the graphs of $x^2-y^2=3$ and $y^2-x^2=3$ using the same axes for both graphs.

## Problems involving quadratic equations

*Example.*—A cylindrical cup is made of 35 sq. in. of thin sheet metal. If its height is 4 in., find its radius to the nearest $\frac{1}{100}$th in.

If the radius is $r$ in., then the area of the curved surface is $2\pi r\times 4$ sq. in. and the area of the base is $\pi r^2$ sq. in.

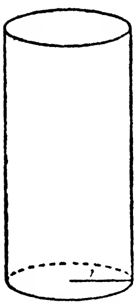




Fig. 21.

$$\therefore\ \pi r^2+8\pi r=35$$

$$\therefore\ r^2+8r=\frac{35}{\pi}=11{\cdot}14$$

$$\therefore\ (r+4)^2=11{\cdot}14+16=27{\cdot}14$$

$$\therefore\ r+4=\pm\sqrt{27{\cdot}14}=\pm 5{\cdot}21$$

$$\therefore\ r=1{\cdot}21,$$

the negative value being clearly an impossible one for the radius of the cup.

*Example.*—Solve the same problem as in the above example, given that the area of the metal is S sq. in. and the height of the cup is $h$ in.

If the radius is $r$ in., the area of the curved surface is $2\pi rh$ and the area of the base is $\pi r^2$ :

$$\therefore \pi r^2 + 2\pi rh = S$$

$$\therefore r^2 + 2rh = \frac{S}{\pi}$$

$$\therefore (r+h)^2 = \frac{S}{\pi} + h^2$$

$$\therefore r + h = \pm\sqrt{\frac{S}{\pi} + h^2}$$

$$\therefore r = \sqrt{\frac{S}{\pi} + h^2} - h,$$

the other root being negative.

*Example.*—If a stone is thrown vertically up under gravity at $u$ ft./sec. its height $s$ ft. after $t$ sec. is given by $s = ut - \frac{1}{2}gt^2$ (neglecting air resistance). Given that $u = 60$, $g = 32$, find when the stone is 40 ft. high.

Substituting the given values :

$$40 = 60t - 16t^2$$

$$\therefore 16t^2 - 60t = -40$$

$$\therefore t^2 - \tfrac{60}{16}t = -\tfrac{40}{16}$$

$$\therefore t^2 - \tfrac{15}{4}t = -\tfrac{40}{16}$$

$$\therefore (t - \tfrac{15}{8})^2 = -\tfrac{40}{16} + \tfrac{225}{64} = \frac{-160 + 225}{64} = \tfrac{65}{64}$$

$$\therefore t - \tfrac{15}{8} \simeq \pm\frac{8 \cdot 06}{8}$$

$$\therefore t \simeq \frac{23 \cdot 06}{8} \text{ or } \frac{6 \cdot 94}{8}$$

$$\therefore t \simeq 2 \cdot 88 \text{ or } 0 \cdot 87.$$

In this example both the answers have a meaning; the smaller answer, 0·87, means that the stone takes 0·87 sec. to get to a height of 40 ft. on the way up, and $t = 2 \cdot 88$ means

that the stone is again at a height of 40 ft. at a time 2·88 sec. after it was thrown up, that is after it has risen to its highest position and fallen back again.

It should be noted from these two examples that in a practical problem either one or both roots of the quadratic equation may apply to the problem.

## Exercise XII

1. From a rectangular sheet of metal 17 in. long, 10 in. wide, a strip $x$ in. wide is cut off all round and the area of the remainder is the same as that of a square of side 12 in. Find $x$.

2. A sheet of metal is to be cut in the shape shown, so as to have an area of 30 sq. in. If it is 10 in. long find the radius of the semicircle.

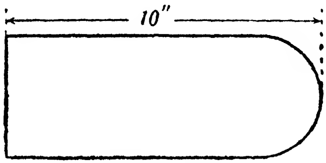



FIG. 22.

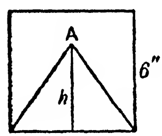



FIG. 23.

3. If a body is thrown vertically upwards under gravity at 100 ft./sec. its height $s$ ft. after $t$ sec. is given by $s = 100t - 16t^2$. At what time is it (a) 136 ft. high, (b) 100 ft. high.

4. If a beam of length $l$ ft. is clamped horizontally at each end, the bending moment at $x$ ft. from one end is $\frac{w}{6\text{EI}}(6x^2 - 6lx + l^2)$. Show that the bending moment is zero at the points given by $x = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)l$. Find the values of $x$ to the nearest inch if $l = 12$.

5. From a square with sides of length 6 in. (Fig. 23) an isosceles triangle, which has a side of the square as base and vertex at A, is cut out. If the centroid of the remaining area is at A prove that $h^2 - 18h + 54 = 0$. Find $h$ from this equation.

6. A room is 12 ft. by 10 ft. Find $x$ so that when the length and breadth of the room are each increased by $x$ ft. the area of the room is increased 50%. Also find $x$ if the length and breadth are initially $a$ ft. and $b$ ft. and the area is increased $r$%.

7. If the bob of a conical pendulum or governor in which the string or rod has a length $l$ ft. describes a circle at $v$ ft./sec. the depth $h$ ft. of the bob beneath the point of suspension is given by $\frac{v^2}{g}=\frac{l^2-h^2}{h}$. Find $h$ if $l=\frac{1}{2}$, $v=2$, $g=32$. Also find a general expression for $h$ in terms of $g$, $v$ and $l$.

8. Find the radius of a solid cylinder 10 in. long if its total surface area is 600 sq. in.

9. A force W lb. wt. stretches a spring whose unstretched length is $l$ ft. by $a$ ft. If one end of the spring is fixed and a weight W lb. wt. attached to the other end is made to describe a horizontal circle at $v$ ft./sec., the string is stretched $x$ ft. where

$$\frac{v^2}{g(l+x)}=\frac{x}{a}.$$

If $g=32$, $l=2$, $a=\frac{1}{10}$, find ($a$) $v$ when $x=\frac{1}{5}$, ($b$) $x$ when $v=20$.

10. If a rod of length $l$ in. is supported at two points $a$ in. from each end, its ends are horizontal if $\frac{wl}{4}\left(\frac{l}{2}-a\right)^2=\frac{wl^3}{48}$. Find $a$ in terms of $l$.

11. The impedance Z ohms of a circuit containing a resistance R ohms, inductance L henries, capacity C farads, when the frequency of the oscillations is $n$ per sec., is given by

$$Z=\sqrt{R^2+\left(2\pi nL-\frac{1}{2\pi nC}\right)^2}.$$

Make L the subject of this formula. If $n=50$, $R=15$, $C=10^{-4}$, show that there are two values of L which make $Z=20$, but only one value which will make $Z=100$. Find these values.

12. If a train goes from one station to another $d$ ft. away in $t$ sec. by accelerating at $f$ ft./sec.$^2$, then travelling at a uniform speed $v$ ft./sec. and finally coming to rest with a deceleration of $f'$ ft./sec.$^2$, $v^2\left(\frac{1}{f}+\frac{1}{f'}\right)-2tv+2d=0$. Calculate the value of $v$ if $f=\frac{1}{2}$, $f'=1$, $t=180$ and the distance between the stations is 1 mile. Find a general expression for $v$ in terms of $f$, $f'$, $t$ and $d$.

13. If a train of length $l$ ft. passes over a bridge the maximum bending moment of the bridge is greatest when the front of the train has reached a point $x$ ft. from one end given by $w_t x^2=2lw_b\left(\frac{l}{2}-x\right)$, where $w_t$ and $w_b$ are the weights per unit length of train and bridge respectively. Find $x$.

**Equal roots and imaginary roots**

*Example.*—Draw the graph of $y = x^2 - 4x$ and use it to solve the equations $x^2 - 4x = 2$, $x^2 - 4x + 2 = 0$, $x^2 - 4x + 4 = 0$, $x^2 - 4x + 6 = 0$.

The graph in Fig. 24 is drawn from the following table :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ .. | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $x^2$ .. | 4 | 1 | 0 | 1 | 4 | 9 | 16 | 25 |
| $-4x$ .. | 8 | 4 | 0 | $-4$ | $-8$ | $-12$ | $-16$ | $-20$ |
| $y$ .. | 12 | 5 | 0 | $-3$ | $-4$ | $-3$ | 0 | 5 |

GRAPH OF $y = x^2 - 4x$

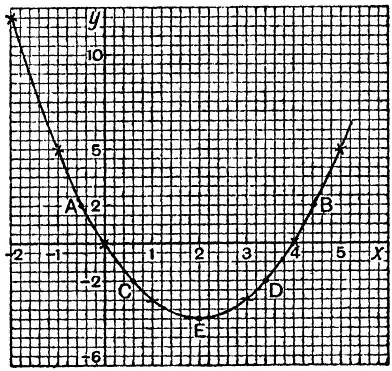



FIG. 24.

$y = 2$, that is $x^2 - 4x = 2$, at the points A and B at which $x \simeq -0{\cdot}4$ and $4{\cdot}5$.

$\therefore$ the roots of $x^2 - 4x = 2$ are $x \simeq -0{\cdot}4$ or $4{\cdot}5$.

By completing the square, these roots are $2 \pm \sqrt{6}$, or $-0{\cdot}4495$ and $4{\cdot}4495$.

In the same way the equation $x^2 - 4x + 2 = 0$ can be written $x^2 - 4x = -2$, and so the roots of the equation are given by the values of $x$ at C and D where $y = -2$. These are $0{\cdot}6$ and $3{\cdot}4$ nearly. $x^2 - 4x + 4 = 0$ can be written $x^2 - 4x = -4$,

and $y = -4$ only at the one point E at which $x=2$. Although there is only one point giving this value of $y$ it is clear that if, starting at CD, we draw horizontal lines lower and lower they will cut the curve in *two points* which gradually approach closer and closer together. For this reason, when the points actually come together at $x=2$, we say that the equation has two roots both equal to 2, instead of saying that it has one root.

If $x^2-4x+6=0$, then $x^2-4x=-6$, and there are no points on the graph of $y=x^2-4x$ at which $y=-6$. Hence this equation has no roots which are ordinary numbers like $3\frac{1}{2}$, 6·8, etc. If, however, we solve the equation by completing the square, we get :

$$x^2-4x+4=4-6=-2.$$

$$\therefore (x-2)^2=-2$$

$$\therefore x-2=\pm\sqrt{-2}=\pm\sqrt{2}\sqrt{-1}$$

$$\therefore x-2=\pm 1{\cdot}41\sqrt{-1}$$

$$\therefore x=2+1{\cdot}41\sqrt{-1} \text{ or } 2-1{\cdot}41\sqrt{-1}.$$

We shall see later that use is made of numbers like these in practical problems in physics and engineering, but for the present we merely note that the roots of a quadratic equation are not always numbers like $\frac{2}{3}$, 3·75, $-6$, which can be marked on a graph.

## Solution of equations of higher degree

Generally such equations have to be solved graphically. However, we can sometimes solve them by using factors.

*Example.*—Solve the equation $x^3-8x-8=0$.

The factors of 8 are $\pm1$, $\pm2$, $\pm4$, $\pm8$, so we try these values of $x$ to see if any one is a root.

$x=1$ makes $x^3-8x-8=1-8-8$; which is not 0.

$x=-1$ makes $x^3-8x-8=-1+8-8$, which is not 0.

$x=2$ makes $x^3-8x-8=8-16-8$, which is not 0.

$x=-2$ makes $x^3-8x-8=-8+16-8$, which *does* equal 0.

Hence $x = -2$ is a root of the equation. This suggests that $(x+2)$ is a factor of $(x^3 - 8x - 8)$, so we try this by long division :

$$\begin{array}{r|l|l} x+2 & x^3 - 8x - 8 & x^2 - 2x - 4 \\ & x^3 + 2x^2 & \\ \hline & \quad -2x^2 - 8x - 8 & \\ & \quad -2x^2 - 4x & \\ \hline & \qquad\quad -4x - 8 & \\ & \qquad\quad -4x - 8 & \\ \hline \end{array}$$

$$\therefore\ x^3 - 8x - 8 = (x+2)(x^2 - 2x - 4)$$

$$\therefore\ x^3 - 8x - 8 = 0 \text{ when}$$

$$\begin{array}{l|l} x + 2 = 0 & \text{or}\quad x^2 - 2x - 4 = 0 \\ \therefore\ x = -2 & \qquad x = 1 \pm \sqrt{5} \end{array}$$

$$\therefore\ x = -2 \text{ or } 1 + \sqrt{5} \text{ or } 1 - \sqrt{5}$$

Hence $x = -2$ or $3{\cdot}236$ or $-1{\cdot}236$ (approx.).

## Graphical solution of an equation

If an equation cannot be solved by using factors we solve it graphically as in the following example.

*Example.*—Solve the equation $x^3 - 3x^2 - 9x + 10 = 0$.

Write $y = x^3 - 3x^2 - 9x + 10$; then the required values of $x$ make $y = 0$, which means that they are given by the values of $x$ at the points where the graph of $y = x^3 - 3x^2 - 9x + 10$ cuts the axis $x'0x$. The graph is plotted in Fig. 25 from the following table.

| $x$ .. | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|
| $x^3$ .. | $-27$ | $-8$ | $-1$ | $0$ | $1$ | $8$ | $27$ | $64$ | $125$ |
| $-3x^2$ .. | $-27$ | $-12$ | $-3$ | $0$ | $-3$ | $-12$ | $-27$ | $-48$ | $-75$ |
| $-9x$ .. | $27$ | $18$ | $9$ | $0$ | $-9$ | $-18$ | $-27$ | $-36$ | $-45$ |
| $+10$ .. | $10$ | $10$ | $10$ | $10$ | $10$ | $10$ | $10$ | $10$ | $10$ |
| $y =$ sum | $-17$ | $8$ | $15$ | $10$ | $-1$ | $-12$ | $-17$ | $-10$ | $15$ |

GRAPH OF $y=x^3-3x^2-9x+10$

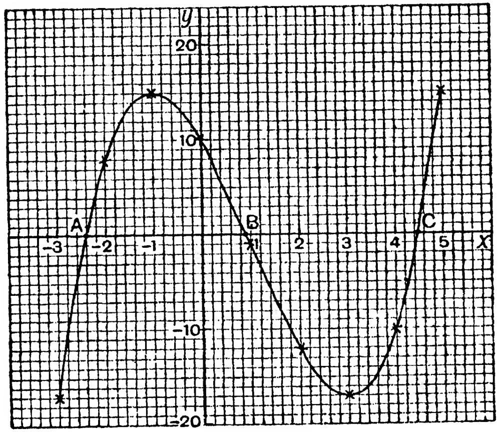



FIG. 25.

The graph cuts $x'0x$ at A, B, C, and the values of $x$ at these points are $-2{\cdot}4$, $0{\cdot}9$, $4{\cdot}5$. Hence, $y=0$, when $x=-2{\cdot}4$ or $0{\cdot}9$ or $4{\cdot}5$ approximately.

The equation can, however, be solved more quickly as follows. Write it :

$$x^3-3x^2=9x-10.$$

Plot the graphs of $y_1=x^3-3x^2$, and $y_2=9x-10$. This is done in Fig. 26 from the tables below :

$$y_1=x^3-3x^2.$$

| $x$ .. | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|
| $x^3$ .. | $-27$ | $-8$ | $-1$ | $0$ | $1$ | $8$ | $27$ | $64$ | $125$ |
| $-3x^2$ .. | $-27$ | $-12$ | $-3$ | $0$ | $-3$ | $-12$ | $-27$ | $-48$ | $-75$ |
| $y_1$ .. | $-54$ | $-20$ | $-4$ | $0$ | $-2$ | $-4$ | $0$ | $16$ | $50$ |

$$y_2=9x-10$$

| $x$ .. | $-3$ | $0$ | $5$ |
|---|---|---|---|
| $y_2$ .. | $-37$ | $-10$ | $35$ |

These graphs intersect in the points P, Q, R. At each of these points the ordinates are equal, that is $y_1 = y_2$, and hence $x^3 - 3x^2 = 9x - 10$.

Therefore, the values of $x$ at P, Q, R, which are nearly $-2{\cdot}4$, $0{\cdot}9$, $4{\cdot}5$ are the roots of the equation.

GRAPHS OF $y_1 = x^3 - 3x^2$ AND $y_2 = 9x - 10$

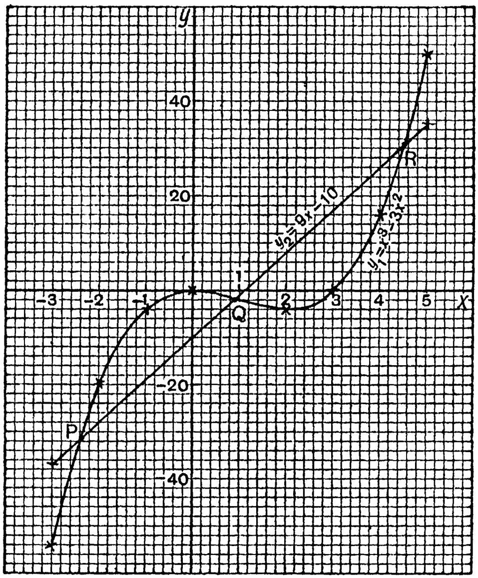



FIG. 26.

To find any one of these roots more accurately we plot the two graphs to a much larger scale near the point of intersection as on p. 63.

The reason this method is quicker than the first is that the graph of $y_2 = 9x - 10$ is a straight line, and so can be drawn by plotting two points only (though three should be plotted to ensure accuracy).

## Exercise XIII

**1.** Draw the graph of $y=x^2-8x$ and use it to find the roots of the equations (*a*) $x^2-8x=4$, (*b*) $x^2-8x+16=0$. Check your answers by calculation.

**2.** Show that the equation $x^2+6x+10=0$ has no real roots. What value must $a$ have for the equation $x^2+6x+a=0$ to have equal roots ?

**3.** Show that $x=1$ is a root of the equation $x^3+2x^2-x-2=0$ and find the other roots.

**4.** One of the numbers 1, 2, 3, is a root of the equation $x^3+5x^2-2x-24=0$. Find which it is, and find the other roots of the equation.

**5.** The point of maximum deflection of a beam is at a distance $x$ from one end where $\frac{x^4}{24}-\frac{l^2x^2}{12}+\frac{7l^4}{360}=0$. Find $x$ in terms of $l$.

**6.** Find the three values of $x$ which make $x^3-16x=10$: (*a*) by drawing the graph of $y=x^3-16x-10$ from $x=-4$ to $5$; (*b*) by drawing the graphs of $y=x^3$ and $y=16x+10$ from $x=-4$ to 5.

**7.** Show by a graph that the equation $2x^3+9x-8=0$ has one root. Find its value to 3 sig. fig.

**8.** The volume V of a spherical segment of height $h$ of a sphere of radius $r$ is given by $V=\frac{1}{3}\pi h^2(3r-h)$. If $r=10$ find $h$ so that the volume of the segment is a quarter of the volume of the sphere. [$h$ is the distance a sphere of specific gravity $\frac{1}{4}$ is immersed when it is floating in water.]

**9.** If a beam 12 ft. long of uniform cross-section is clamped horizontally at one end and loaded at the free end, the deflection at a point $x$ ft. from the clamped end is half the deflection at the other end when $x^3-36x^2+1728=0$. Find the value of $x$ to the nearest inch.

# CHAPTER IV

## LOGARITHMS

The name logarithm is derived from the Greek words λογος = reckoning, αριθμος = number. Although there had been several attempts during the sixteenth century to find some way of replacing multiplication by addition, John Napier, of Merchiston (near Edinburgh), was the first person to construct a table of [illegible]. This table, which was published in 1614, was a table of seven-figure logarithms of the sines of angles at every minute from 0° to 90°. The base used for the table was not 10, but $\frac{1}{e}$, where $e$ is a number, nearly 2·7183, which has since become of increasing importance in applications of mathematics.

Henry Briggs, professor of geometry at Gresham College, London, was so interested in Napier's discovery that he went to stay with him in 1615 and 1616 to discuss the calculation of logarithm tables. They both had the plan of making a table of logarithms to base 10, but Napier died in April, 1617. In the same year Briggs published a table of logarithms to base 10 of all numbers from 1 to 1000, and in 1624 he published a more comprehensive table which contained the logarithms of all numbers from 1 to 20,000 and from 90,000 to 100,000 to fourteen decimal places. Thus Briggs's larger table was not complete, but it was completed as soon after as 1628 by Adrian Vlacq, of Gouda in Holland, who published an eight-figure table of the logarithms to base 10 of all numbers from 1 to 100,000, and most of the tables published since have been based upon it.

In 1620 Jobst Bürgi, a Swiss mathematician, published a table of anti-logarithms in Prague. It is generally believed that the calculation of this table was quite independent of the work of Napier and Briggs.

Edmund Gunter was the first person to realize (in 1620)

that sliding rules with logarithmic scales could be used for multiplication, but the first slide rule, of the type we know to-day, was made by Robert Bissaker in 1654; it is in the Science Museum, South Kensington.

**The graph of $2^x$**

The two following tables show the values of $2^x$ when $x=0$, $\frac{1}{2}$, 1, . . . 3, and when $x=-\frac{1}{2}$, $-1$, . . . $-3$.

| $x$ | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 |
|---|---|---|---|---|---|---|---|
| $2^x$ | 1 | $\sqrt{2}=1{\cdot}414$ | 2 | $2\sqrt{2}=2{\cdot}828$ | 4 | $4\sqrt{2}=5{\cdot}657$ | 8 |

| $x$ | $-\frac{1}{2}$ | $-1$ | $-1\frac{1}{2}$ | $-2$ | $-2\frac{1}{2}$ | $-3$ |
|---|---|---|---|---|---|---|
| $2^x$ | $\frac{1}{\sqrt{2}}=0{\cdot}707$ | $\frac{1}{2}=0{\cdot}5$ | $\frac{1}{2\sqrt{2}}=0{\cdot}354$ | $\frac{1}{2^2}=0{\cdot}25$ | $\frac{1}{4\sqrt{2}}=0{\cdot}177$ | $\frac{1}{2^3}=0{\cdot}125$ |

GRAPH OF $2^x$

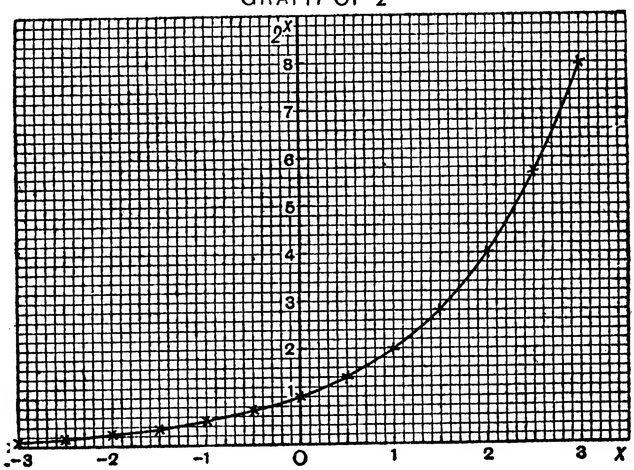



FIG. 27.

From the tables the points in Fig. 27 are plotted and joined by a smooth curve. It can easily be verified that any point obtained by giving another value to $x$, say $\frac{3}{4}$, lies on the curve. The student should find the values of $2^{\frac{1}{4}}$, $2^{\frac{3}{4}}$, $2^{\frac{5}{4}}$, etc., by finding the square roots of $2^{\frac{1}{2}}$, $2^{\frac{3}{2}}$, $2^{\frac{5}{2}}$, and verify that the points so found lie on the graph. It is very important to notice that as $x$ varies the values of $2^x$ do give points lying on a smooth curve. Thus $2^{0\cdot1}$ is slightly larger than $2^0$, which is 1, and $2^{2\cdot4}$ is slightly smaller than $2^{2\cdot5}$, which is 5·657.

The graph of $y=a^x$, where $a$ is a number greater than 1, is similar in shape to the graph of $y=2^x$. The student should draw the graphs of $y=2^x$, $y=3^x$, $y=4^x$, using the same axes and scales for each graph.

## Logarithms

*If* $y=2^x$ *we call* $x$ *the logarithm of* $y$ *to the base* 2, *and write this* $x=\log_2 y$. This clearly means that $x=\log_2 (2^x)$.

From the graph of $y=2^x$ we find that, when $y=0\cdot8$, $x \simeq -0\cdot32$, in other words, $0\cdot8 \simeq 2^{-0\cdot32}$ or $\log_2 0\cdot8 \simeq -0\cdot32$. In the same way, $3\cdot6 \simeq 2^{1\cdot85}$, or $\log_2 3\cdot6 \simeq 1\cdot85$ :

$$\therefore\ 3\cdot6 \times 0\cdot8 = 2^{1\cdot85} \times 2^{-0\cdot32} = 2^{1\cdot85-0\cdot32},$$

which means that

$$\log_2 (3\cdot6 \times 0\cdot8) = 1\cdot85 - 0\cdot32 = \log_2 3\cdot6 + \log_2 0\cdot8.$$

This shows that a product can be found by addition of logarithms; multiplication is replaced by addition. Herein lies the importance of logarithms.

Since $2^x$ is positive whether $x$ is positive or negative, we cannot find any value of $x$ to make $2^x$ negative; in other words, a negative number has no logarithm.

Similarly, if $y=a^x$, where $a$ is any positive number, we call $x$ the logarithm of $y$ to the base $a$ and write this $x=\log_a y$.

**Logarithms to the base 10**

By the definition above :

$$\log_{10} 10^n = n$$

$$\therefore \log_{10} 10 = \log_{10} 10^1 = 1$$

$$\log_{10} 100 = \log_{10} 10^2 = 2$$

$$\log_{10} 1000 = \log_{10} 10^3 = 3$$

Similarly :

$$\log_{10}\left(\frac{1}{10}\right) = \log_{10} 10^{-1} = -1$$

$$\log_{10}\left(\frac{1}{10^2}\right) = \log_{10} 10^{-2} = -2$$

and so on.

Therefore :

$$\log_{10} 25 = \log_{10}(10 \times 2{\cdot}5) = \log_{10} 10 + \log_{10} 2{\cdot}5$$
$$= 1 + 0{\cdot}3979 = 1{\cdot}3979$$

$$\log_{10} 250 = \log_{10}(10^2 \times 2{\cdot}5) = \log_{10} 10^2 + \log_{10} 2{\cdot}5$$
$$= 2 + 0{\cdot}3979 = 2{\cdot}3979$$

Similarly :

$$\log_{10} 0{\cdot}25 = \log_{10}(10^{-1} \times 2{\cdot}5) = \log_{10} 10^{-1} + \log_{10} 2{\cdot}5$$
$$= -1 + 0{\cdot}3979 = \bar{1}{\cdot}3979$$

$$\log_{10} 0{\cdot}025 = \log_{10}(10^{-2} \times 2{\cdot}5) = \log_{10} 10^{-2} + \log_{10} 2{\cdot}5$$
$$= -2 + 0{\cdot}3979 = \bar{2}{\cdot}3979$$

Thus, using the base 10, the logarithms of all numbers formed by the same digits have the same positive decimal part or *mantissa*, the whole number part or *characteristic* being determined only by the position of the decimal point. This is not true for any other base, and this is the reason that logarithms to base 10 are used in nearly all calculations. We shall see later that there are a few problems in which it is convenient to use the number $e$ as base. When using the base 10 the suffix indicating the base is usually omitted.

**Rules of logarithms**

Let $a$ and $b$ be any two positive numbers and let

$$a = 10^m, \quad b = 10^n.$$

Then,

$$\log a = m, \quad \log b = n$$

$$\therefore \log ab = \log (10^m \times 10^n) = \log 10^{m+n} = m + n$$

$$\therefore \underline{\log ab = \log a + \log b.}$$

In the same way,

$$\log \frac{a}{b} = \log \left(\frac{10^m}{10^n}\right) = \log 10^{m-n} = m - n$$

$$\therefore \underline{\log \frac{a}{b} = \log a - \log b.}$$

Also, if $r$ is any number, positive or negative,

$$\log a^r = \log (10^m)^r = \log 10^{mr} = mr$$

$$\therefore \underline{\log a^r = r \log a.}$$

By using the formulæ underlined above with a table of logarithms we can calculate products, quotients and powers by addition and subtraction.

In carrying out a calculation it is often helpful to give a name, say $x$, to the number to be calculated, and then to write down an equation for $\log x$ before actually tabulating the logarithm.

*Example.*—Calculate $\dfrac{27{\cdot}12 \times (0{\cdot}7065)^2}{0{\cdot}004729 \times (987{\cdot}6)^{\frac{1}{3}}}$.

If this number is $x$,

$$\begin{aligned} \log x &= \log (\text{numerator}) - \log (\text{denominator}) \\ &= \{\log 27{\cdot}12 + \log (0{\cdot}7065)^2\} - \{\log 0{\cdot}004729 \\ &\qquad\qquad + \log (987{\cdot}6)^{\frac{1}{3}}\} \\ &= \{\log 27{\cdot}12 + 2 \log 0{\cdot}7065\} - \{\log 0{\cdot}004729 \\ &\qquad\qquad + \tfrac{1}{3} \log 987{\cdot}6\} \end{aligned}$$

This equation shows us what logarithms have to be tabulated. The actual calculation is shown below.

| *Number* | *Logarithm* | |
|---|---|---|
| 27·12 | 1·4333 | |
| $(0{\cdot}7065)^2$ | $2 \times \bar{1}{\cdot}8491 = \bar{1}{\cdot}6982$ | |
| | 1·1315 | 1·1315 |
| 0·004729 | $\bar{3}{\cdot}6747$ | |
| $(987{\cdot}6)^{\frac{1}{3}}$ | $\frac{1}{3}(2{\cdot}9946) = 0{\cdot}9982$ | |
| | $\bar{2}{\cdot}6729$ | $\bar{2}{\cdot}6729$ |
| 287·5 | | 2·4586 |

*Ans.* 287·5.

*Example.*—Find the value of $(0{\cdot}009162)^{\frac{1}{4}}$.

If $x = (0{\cdot}009162)^{\frac{1}{4}}$

$\log x = \frac{1}{4} \log 0{\cdot}009162 = \frac{1}{4}(\bar{3}{\cdot}9620)$.

Therefore, because $\bar{3} = -3 = -4 + 1 = \bar{4} + 1$,

$\log x = \frac{1}{4}(\bar{4} + 1{\cdot}9620) = \bar{1} + 0{\cdot}4905 = \bar{1}{\cdot}4905$

$\therefore\ x = 0{\cdot}3094$.

*Example.*—Find the value of $(0{\cdot}6173)^{3\cdot15}$.

If $x = (0{\cdot}6173)^{3\cdot15}$

$\log x = 3{\cdot}15 \log 0{\cdot}6173 = 3{\cdot}15\ (\bar{1}{\cdot}7905)$.

Now $\bar{1}{\cdot}7905 = -1 + 0{\cdot}7905 = -0{\cdot}2095$.

$\therefore \log x = -3{\cdot}15 \times 0{\cdot}2095.$
$= -0{\cdot}6600$
$= -1 + 0{\cdot}3400$
$= \bar{1}{\cdot}3400$
$\therefore\ x = 0{\cdot}2188.$

| *No.* | *Log* |
|---|---|
| 3·15 | 0·4983 |
| 0·2095 | $\bar{1}{\cdot}3212$ |
| 0·6600 | $\bar{1}{\cdot}8195$ |

*Example.*—Calculate the values of $(1-t)e^{-t}$ where $e=2\cdot7183$, when $t=0\cdot6124$ and when $t=1\cdot7329$.

Calling the given expression $y$, when $t=0\cdot6124$,

$$y=0\cdot3876\times2\cdot7183^{-0\cdot6124}$$

| | *No.* | *Log* |
|---|---|---|
| $\therefore \log y=\log 0\cdot3876-0\cdot6124\log 2\cdot7183$ | | |
| $=\bar{1}\cdot5884-0\cdot6124\times0\cdot4343$ | 0·6124 | $\bar{1}\cdot7871$ |
| $=\bar{1}\cdot5884-0\cdot2661$ | 0·4343 | $\bar{1}\cdot6378$ |
| $=\bar{1}\cdot3223$ | | |
| $\therefore y=0\cdot2100.$ | 0·2661 | $\bar{1}\cdot4249$ |

When $t=1\cdot7329$,

$$y=-0\cdot7329\times2\cdot7183^{-1\cdot7329}.$$

Since $y$ is now a negative number we cannot take logs of this equation as it stands, so we write :

$$z=0\cdot7329\times2\cdot7183^{-1\cdot7329}$$

| | *No.* | *Log* |
|---|---|---|
| Then $\log z=\log 0\cdot7329-1\cdot7329\log 2\cdot7183$ | | |
| $=\bar{1}\cdot8650-1\cdot7329\times0\cdot4343$ | 1·7329 | 0·2388 |
| $=\bar{1}\cdot8650-0\cdot7526$ | 0·4343 | $\bar{1}\cdot6378$ |
| $=\bar{1}\cdot1124$ | | |
| $\therefore z=0\cdot1295$ | 0·7526 | 1·8766 |
| $\therefore y=-z=-0\cdot1295$ | | |

*Example.*—Make a table of the values of $8\cdot43\times2\cdot6^{-x}$ for $x=-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3$, and hence draw the graph of $y=8\cdot43\times2\cdot6^{-x}$. Use the graph to find approximately (i) the value of $y$ when $x=-0\cdot85$, (ii) the value of $x$ when $y=40$.

By the rules of logarithms :

$$\log y=\log 8\cdot43-x\log 2\cdot6$$
$$=0\cdot9258-0\cdot4150x.$$

From this equation the following table is constructed to give $\log y$, and then $y$ is found from the antilogarithm tables :

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| log. 8·43 .. | 0·9258 | 0·9258 | 0·9258 | 0·9258 | 0·9258 | 0·9258 | 0·9258 |
| $-0\cdot4150x$ .. | 1·2450 | 0·8300 | 0·4150 | 0 | −0·4150 | −0·8300 | −1·2450 |
| log $y$ .. .. | 2·1708 | 1·7558 | 1·3408 | 0·9258 | 0·5108 | 0·0958 | $\bar{1}\cdot6808$ |
| $y$ .. .. | 148·2 | 56·90 | 21·92 | 8·429 | 3·242 | 1·247 | 0·480 |

The graph is shown in Fig. 28, and from it we find :

(i) $y \simeq 18 \cdot 7$ when $x = -0 \cdot 85$ at the point A.

(ii) $x \simeq -1 \cdot 65$ when $y = 40$ at the point B.

GRAPH OF $y = 8{\cdot}43 \times 2{\cdot}6^{-x}$

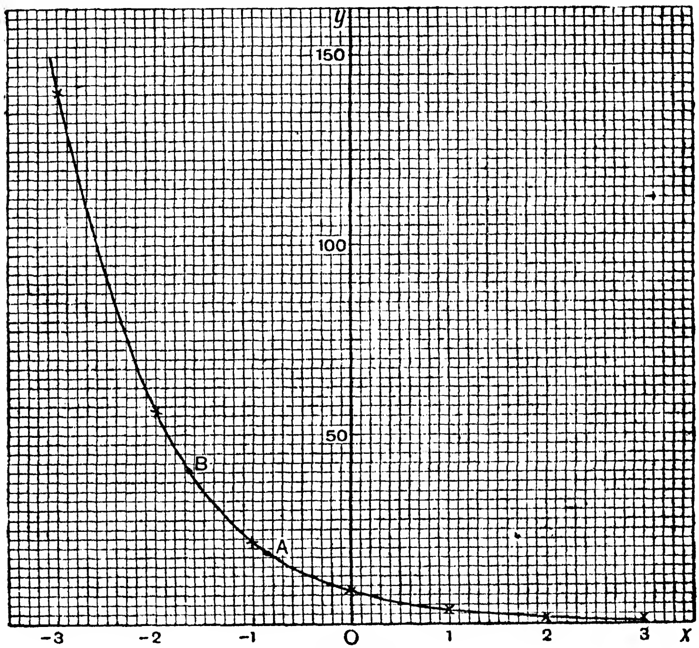



FIG. 28.

## Exercise XIV

Calculate the values of :

1. $\dfrac{237{\cdot}9 \times 6{\cdot}248}{9832}$.

2. $\dfrac{0{\cdot}2791}{0{\cdot}04125}$.

3. $\dfrac{0{\cdot}9328 \times 4{\cdot}197^2}{0{\cdot}2196^3}$.

4. $\dfrac{\sqrt{7{\cdot}916} \times 43{\cdot}21}{1760}$.

5. $\dfrac{\sqrt[3]{178\cdot2}}{\sqrt{278\cdot1}}$.

6. $\dfrac{46\cdot19 \times 0\cdot001753}{\sqrt[3]{195\cdot7}}$.

7. $\left(\dfrac{47\cdot23 \times 0\cdot1972}{2\cdot456}\right)^2$.

8. $\sqrt{\dfrac{47\cdot23 \times 0\cdot1792}{2\cdot4256}}$.

9. $0\cdot735^{\frac{1}{3}}$.

10. $0\ 0009^{\frac{1}{5}}$.

11. $432\cdot7^{1\cdot427}$.

12. $0\cdot6459^{2\cdot56}$.

13. $83\cdot9^{-1\cdot2}$.

14. $0\cdot8731^{-4}$.

15. $\dfrac{0\cdot0437}{(0\cdot9436)^{\frac{1}{3}}}$.

16. $0\cdot9712 \times 8\cdot7^{0\cdot431}$.

17. $\sqrt{\sin 81^\circ\ 15'}$.

18. $\dfrac{\text{Cos } 23^\circ\ 4' \cos 41^\circ\ 52'}{\cos 35^\circ\ 6'}$.

19. $\dfrac{145\cdot2 \sin 59^\circ\ 10'}{\sin 75^\circ\ 39'}$.

20. $\dfrac{\tan 37^\circ\ 23'}{\tan 64^\circ\ 19'}$.

21. Draw a graph of $y=3^x$ from $x=-2$ to 2 using values of $x$ at intervals of $\frac{1}{2}$. From your graph read off the values of $\log_3 6$ and $\log_3 0\cdot5$.

22. Draw a graph of $y=10^x$ from $x=0$ to 1 using values of $x$ at intervals of $\frac{1}{4}$ and a table of square roots. From it read off the values of $10^{0\cdot3}$, $10^{0\cdot7}$, $\log_{10} 2$, $\log_{10} 6$, $\log_{10} 9$. Deduce the values of $\log_{10} 0\cdot2$, $\log_{10} 600$ and $\log_{10} 0\cdot09$.

Prove that :

23. $\log \dfrac{p}{q} = \log p - \log q$.

24. $\log \dfrac{1}{k} = -\log k$.

25. $\log \sqrt[n]{\dfrac{a}{b}} = \dfrac{\log a - \log b}{n}$.

26. $\log \left(\dfrac{1}{r}\right)^{-n} = n \log r$.

27. Calculate the values of $50x^{0\cdot8}$ when $x$ is $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, 2, 3 and 4, and draw a graph of this function from $x=0$ to 4. From the graph find at what values of $x$ the function has the values 40 and 120 respectively.

28. Calculate the values of $4\cdot75 \times (0\cdot943)^x$ when $x$ is $-2$, $-1$, to 1, 2 and 3, and draw a graph of $y=4\cdot75 \times (0\cdot943)^x$ from $x=-2$ 0, 3. From the graph find the value of $y$ when $x=1\cdot7$ and the value of $x$ when $y=4$.

29. Using logarithms to base 10 calculate the values of $e^x$, where $e=2\cdot7183$, when $x$ is $-2$, $-1$, 0, 1, 2, and 3. Deduce from these the values of $e^{-x}$ and $\frac{1}{2}(e^x+e^{-x})$, and draw the graphs

of these three functions on the same axes from $x = -2$ to 3. Find the values of $x$ which make the three functions each equal to 6.

30. Draw a graph of $y = 0{\cdot}004x^{5{\cdot}5}$ from $x = 4$ to $x = 6$, and find from it the value of $x$ [to 3 sig. fig.] which makes $y$ equal to 35.

## Evaluation of technical formulæ

In calculating the value of some quantity from a formula it is advisable to keep to the symbols as long as possible. There is considerable saving in writing one symbol instead of a number of four figures, but this is not the only advantage, for it is easier to trace mistakes and for any one else to follow the argument if symbols are adhered to. For the same reason, one should not generally write 3·1416 for $\pi$, or 1·4142 for $\sqrt{2}$ in a formula. The following examples illustrate the evaluation of several formulæ from engineering and physics.

In any of the following examples a slide rule can be used for every step involving multiplication or division provided that only two significant figures are required in the answer.

*Example.*—The inductance, L henrys, of a closely-wound coil on a cylindrical former of diameter $d$ ft. and length $l$ ft. is given by:

$$\mathrm{L} = \frac{k\pi^2 d^2 n^2}{l} \times 10^{-9},$$

where $n$ is the number of turns, and $k$ is a constant depending on the leakage of magnetic flux. Find the value of L for a coil of 28 turns wound on a former of diameter 7 in. and length 15 in., if $k = 0{\cdot}8$.

Expressing the diameter and length in feet:

$$d = \tfrac{7}{12} = 0{\cdot}5833 \text{ and } l = \tfrac{15}{12} = 1{\cdot}25.$$

Taking logs and omitting the factor $10^{-9}$, which can be inserted later,

$$\begin{aligned}\log\left(\frac{k\pi^2 d^2 n^2}{l}\right) &= \log \pi^2 d^2 n^2 + \log k - \log l \\ &= 2 \log \pi d n + \log k - \log l \\ &= 2\,(\log \pi + \log d + \log n) + \log k - \log l.\end{aligned}$$

| *No.* | | | *Log* |
|---|---|---|---|
| $\pi$ | .. | .. | 0·4971 |
| $d$ | .. | .. | $\bar{1}$·7659 |
| $n$ | .. | .. | 1·4472 |
| $\pi dn$ | .. | .. | 1·7102 |
| $(\pi dn)^2$ | | .. | 3·4204 |
| $k$ | .. | .. | $\bar{1}$·9031 |
| | | | 3·3235 |
| $l$ | .. | .. | 0·0969 |
| 1685 | .. | .. | 3·2266 |

$$\therefore \frac{k\pi^2d^2n^2}{l} = 1685$$

$$\therefore \text{L} \simeq 1685 \times 10^{-9} = 1{\cdot}685 \times 10^{-6}.$$

*Example.*—The period T of small oscillation of a simple pendulum of length $l$ is given by :

$$\text{T} = 2\pi\sqrt{\frac{l}{g}}$$

Calculate the value of $g$ obtained from an experiment in which it is found that the period of oscillation of a pendulum 46 cm. long is 1·36 sec.

First, $g$ should be made the subject of the formula. This can be done before or after taking logs. To make it the subject, first square the given equation :

$$\text{T}^2 = \frac{4\pi^2 l}{g}$$

$$\therefore g\text{T}^2 = 4\pi^2 l$$

$$\therefore g = \frac{4\pi^2 l}{\text{T}^2}$$

$$\therefore \log g = \log 4 + \log \pi^2 + \log l - \log \text{T}^2.$$

Alternatively, taking logs of the given equation:

$$\log T = \log 2 + \log \pi + \tfrac{1}{2}\log l - \tfrac{1}{2}\log g$$
$$\therefore \tfrac{1}{2}\log g = \log 2 + \log \pi + \tfrac{1}{2}\log l - \log T$$
$$\therefore \log g = 2\log 2 + 2\log \pi + \log l - 2\log T.$$

which is the same value of $\log g$ as before, since $2\log 2 = \log 4$, etc. The calculation is set out below:

| *No.* | | *Log* |
|---|---|---|
| 4 | | 0·6021 |
| $\pi^2$ | $2 \times 0{\cdot}4971 =$ | 0·9942 |
| $l$ | | 1·6628 |
| | | 3·2591 |
| $T^2$ | $2 \times 0{\cdot}1335 =$ | 0·2670 |
| $g$ | | 2·9921 |

$$\therefore g = 981{\cdot}9.$$

Note that since $g = \dfrac{4\pi^2 l}{T^2}$ the unit of $g$ is $1\ \dfrac{\text{cm.}}{\text{sec.}^2}$.

*Example.*—When a volume $v_1$ of hydrogen is compressed adiabatically to a volume $v_2$ the relationship between the volumes and the initial and final pressures $p_1$ and $p_2$ is $v_2 = v_1\left(\dfrac{p_1}{p_2}\right)^{\frac{1}{1{\cdot}4074}}$. Calculate $v_2$ if $v_1 = 3{\cdot}45$ ft.$^3$, $p_1 = 15$ lb./in.$^2$ and $p_2 = 200$ lb./in.$^2$.

Since the ratio $p_1/p_2$ is independent of the units of $p_1$ and $p_2$,

$$v_2 = 3{\cdot}45\left(\frac{15}{200}\right)^{\frac{1}{1{\cdot}4074}}$$

gives the volume in cubic feet.

$$\therefore \log v_2 = \log 3{\cdot}45 + \frac{1}{1{\cdot}4074}(\log 15 - \log 200)$$
$$= 0{\cdot}5378 + \frac{1}{1{\cdot}4074}(1{\cdot}1761 - 2{\cdot}3010)$$

$$=0{\cdot}5378-\frac{1{\cdot}125}{1{\cdot}4074}$$
$$=0{\cdot}5378-0{\cdot}7995$$
$$=-1+(1{\cdot}5378-0{\cdot}7995)$$
$$=\bar{1}{\cdot}7383$$

| *No.* | *Log* |
|---|---|
| 1·125 | 0·0511 |
| 1·4074 | 0·1483 |
| 0·7995 | $\bar{1}$·9028 |

$$\therefore\ v_2=0{\cdot}5474.$$

**The use of indices and logarithms in changing the subject of a formula**

Suppose $y$ is to be made the subject of the formula $y^{0\cdot3}=2$. To do this $y^{0\cdot3}$ must be raised to such a power that the result is $y$. This power is $\frac{1}{0\cdot3}$ because,

$$(y^{0\cdot3})^{\frac{1}{0\cdot3}}=y^{0\cdot3\times\frac{1}{0\cdot3}}=y^1=y.$$

Hence, raising both sides of the equation $y^{0\cdot3}=2$ to the power $\frac{1}{0\cdot3}$,

$$y=2^{\frac{1}{0\cdot3}}=2^{\frac{10}{3}}\text{ or }\sqrt[3]{2^{10}}.$$
$$\therefore\ y=\sqrt[3]{1024}\simeq10{\cdot}08.$$

This method can be used to make $y$ the subject of any formula which gives the value of $y^n$, where $n$ is any number. Sometimes logarithms are required to express $y$ in its simplest form.

*Example.*—The volume of water Q cu. ft. discharged per minute through a certain V-shaped notch was found by experiment to be given by

$$Q=2{\cdot}72\ H^{2\cdot53}.$$

Make H the subject of this formula.

Dividing by 2·72, $H^{2\cdot53}=\dfrac{Q}{2\cdot72}$.

Raising both sides to the power $\frac{1}{2\cdot53}$,

$$H^{2\cdot53\times\frac{1}{2\cdot53}}=\left(\frac{Q}{2\cdot72}\right)^{\frac{1}{2\cdot53}}$$

$$\therefore\ H=\left(\frac{Q}{2\cdot72}\right)^{\frac{1}{2\cdot53}}.$$

H is the subject of this formula, but the formula is in a more convenient form if we express H in the form $aQ^n$. Since $\frac{1}{2\cdot53}=0\cdot3953$ and $\frac{1}{2\cdot72}=0\cdot3676$,

$$H=(0\cdot3676Q)^{0\cdot3953}=(0\cdot3676)^{0\cdot3953}Q^{0\cdot3953}.$$

To calculate the numerical coefficient on the right-hand side, let

$$\begin{aligned}a&=(0\cdot3676)^{0\cdot3953}\\ \log a&=0\cdot3953\log 0\cdot3676\\ &=0\cdot3953\times\bar{1}\cdot5654\\ &=0\cdot3953\,(-0\cdot4346)\\ &=-(0\cdot3953\times0\cdot4346)\\ &=-0\cdot1718\\ &=\bar{1}\cdot8282\\ \therefore\ a&=0\cdot6733\end{aligned}$$

| *No.* | *Log* |
|---|---|
| 0·3953 | $\bar{1}\cdot5969$ |
| 0·4346 | $\bar{1}\cdot6381$ |
| 0·1718 | $\bar{1}\cdot2350$ |

Hence $H=0\cdot6733\ Q^{0\cdot3953}$.

## Solution of equations using logarithms

If an unknown quantity occurs in an index we can sometimes find its value by taking logs.

*Example.*—Find $x$ if $2^x=0\cdot09173$.

Since the logarithm of $2^x$ is $x\log 2$, by taking logarithms we get:

$$x\log 2=\log 0\cdot09173$$

$$\therefore x(0\cdot3010) = \bar{2}\cdot9625 = -2 + 0\cdot9625$$
$$= -1\cdot0375$$
$$\therefore x = -\frac{1\cdot0375}{0\cdot3010}$$
$$= -3\cdot446.$$

| *No.* | *Log* |
|---|---|
| 1·0375 | 0·0160 |
| 0·3010 | $\bar{1}$·4786 |
| 3·446 | 0·5374 |

Note that before dividing $\bar{2}\cdot9625$ by $0\cdot3010$ we write $\bar{2}\cdot9625$ as a negative number.

## Exercise XV

**1.** The capacity C of a condenser of $n$ plates is given by:

$$C = \frac{kA(n-1)}{4\pi d \times 9 \times 10^{11}}.$$

Find C if $k = 5\cdot42$, $A = 35$, $n = 19$ and $d = 0\cdot008$.

**2.** The whirling speed, N rev. per min., of a shaft is given by:

$$N = 30\pi\sqrt{\frac{12gEI}{wl^4}}.$$

Find N, if $\pi = 3\cdot142$, $g = 32\cdot2$, $E = 30 \times 10^6$, $w = 0\cdot28$, $l = 24$ and $I = 0\cdot0564$.

**3.** If a current of frequency $n$ per sec. passes through a wire of resistance R ohm and inductance L henrys, the impedance Z is given by $Z = \sqrt{R^2 + 4\pi^2n^2L^2}$. Find the value of Z when $R = 28\cdot7$, $L = 0\cdot00462$, $n = 400$.

**4.** If £P is invested at $r$% compound interest it amounts after $n$ years to £A where $A = P\left(1 + \frac{r}{100}\right)^n$. Find A, if $P = 250$, $r = 4$ and $n = 12$.

**5.** If $y = \frac{Wl^3}{4Ebd^3}$, find E when $y = 0\cdot685$ cm., $W = 20{,}000$ gm., $l = 57\cdot7$ cm., $b = 2\cdot54$ cm., $d = 0\cdot635$ cm. State the units of E.

**6.** The ratio of the tensions $T_1$ and $T_2$ of the parts of a belt on opposite sides of a grooved pulley is given by $\frac{T_1}{T_2} = e^{(\mu\pi/\sin\alpha)}$. Find $T_1$, if $T_2 = 150$, $e = 2\cdot718$, $\mu = 0\cdot3$, $\pi = 3\cdot142$, and $\alpha = 20°$.

7. In the construction of an isochronous governor the proportional change in angular velocity due to a change in the angle which the arms make with the vertical from $\alpha$ to $\beta$ is given by:

$$p=\sqrt{\frac{\cos\alpha}{\cos^3\beta}-\frac{\tan^3\beta}{\tan\alpha}}.$$

Find $p$, if $\alpha=36°$ and $\beta=30°$.

8. The maximum current I ampères which can flow in a cable without causing a rise in temperature of more than 20° F. is given by $I=2{\cdot}6\ A^{0{\cdot}82}$, where A is the total cross-sectional area of all the wires forming the cable in 1000ths of a sq. in. Find the total current that a cable with 19 wires of diameter 0·064 in. can transmit without the temperature rising more than 20° F. Also calculate the least number of wires of diameter 0·083 in. which must be used to carry a current of 200 ampères under the same temperature condition.

9. Steinmetz law for the area of a hysteresis loop is $W=\eta B^x$. Find $\eta$ if $W=4300$, $B=11{,}900$, $x=1{\cdot}6$.

10. The quantity of water flowing over a weir is calculated from the formula $Q=3{\cdot}10\ L^{1{\cdot}02}\ H^{1{\cdot}47}$. Find Q, if $L=18$, $H=10$.

11. The impedance of a circuit is given by:

$$Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2},$$

where $\omega=2\pi n$. Find Z if $R=15$, $L=0{\cdot}1$, $n=50$, $C=80\times10^{-6}$. (Use square and reciprocal tables).

12. The percentage efficiency of a petrol engine is given by:

$$E=100\left[1-\left(\frac{1}{R}\right)^{0{\cdot}25}\right],$$

where R is the expansion ratio. Make a table of the values of E for values of R from 2 to 18 at intervals of 4. Draw a graph of E against R.

13. The relation between the pressure $p$ and volume $v$ of a gas expanding adiabatically is given by $pv^{1{\cdot}41}=C$. Find C if $p=2000$ when $v=5$. Calculate the values of $p$ when $v=2{\cdot}5$, $7{\cdot}5$, $10$, $12{\cdot}5$, and draw a graph of $p$ against $v$ from $v=2{\cdot}5$ to $v=12{\cdot}5$. From it estimate the value of $v$ when $p=1000$.

14. At $t$ sec. after its temperature is 30° C. the temperature of a cooling body, T° C., is given by $T=30e^{-kt}$, where $e=2{\cdot}718$ and $k$ is a constant. If $T=24$ when $t=20$ show that $T=30(0{\cdot}8)^{0{\cdot}05t}$. Calculate the values of T at intervals of 20 sec. from $t=0$ to 120, and draw a graph of T against $t$.

15. The cutting speed of a tool, V ft./min., and the life of the tool, $t$ min., are related by the formula $Vt^{\frac{1}{8}}=C$, where C is a fixed number. It is found that for a certain tool $t=40$ when $V=110$. Calculate the value of C and draw a graph of V against $t$ from $t=20$ to $t=200$. Estimate at what speed the tool will last for 128 min.

16. If $T_2=T_1\left(\frac{v_1}{v_2}\right)^{\gamma-1}$ find the value of $T_2$ when $T_1=3466$, $v_1/v_2=1/5$, and $\gamma=1{\cdot}4$.

Make the letter in brackets after each formula the subject of the formula :

17. $N=188\sqrt{\frac{EI}{wl^4}}$ $(l)$.  18. $vt^{\frac{1}{8}}=C$ $(t)$.

19. $T_2=T_1\left(\frac{v_1}{v_2}\right)^{\gamma-1}$ $(v_1)$.  20. $I=2{\cdot}6A^{0{\cdot}82}$ (A).

21. $E=100\left\{1-\left(\frac{1}{R}\right)^{0{\cdot}25}\right\}$ (R).  22. $pv^n=C$ $(n)$.

23. $Q=3{\cdot}10L^{1{\cdot}02}H^{1{\cdot}47}$ (L).  24. $A=P\left(1+\frac{r}{100}\right)^n$ $(r)$.

25. The power P watts dissipated per sq. cm. from the cooling surface of a cylindrical wire of radius $r$ cm. is given by $P=\frac{\rho i^2}{2\pi^2 r^3}$ where $i$ ampères is the current and $\rho$ ohm/cm.$^3$ the resistivity of the wire. Make (a) $i$, (b) $r$, the subject of this formula. If for a certain material $\rho=1{\cdot}8\times10^{-6}$, calculate the values of $r$ required for $i=1$, 2, 5 and 10, when $P=\frac{1}{10}$.

26. If $pv^{\frac{16}{15}}=a$, and $pv=b$, express $p$ and $v$ in terms of $a$ and $b$. Calculate the value of $p$ if $a=490$ and $b=429$.

### Logarithms to any base; change of base

To find the logarithm of a number $y$ to base $a$ we have to find $x$ so that $y=a^x$. Taking logs of this equation to base 10 :

$$\log_{10} y=\log_{10}(a^x)=x\log_{10} a$$

$$\therefore\ x=\frac{\log_{10} y}{\log_{10} a}.$$

that is $$\log_a y=\frac{\log_{10} y}{\log_{10} a}.$$

It follows that

$$\log_y a = \frac{\log_{10} a}{\log_{10} y} = \frac{1}{\log_a y}.$$

*Example.*—Find $\log_2 3$.

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2} = \frac{0{\cdot}4771}{0{\cdot}3010} = 1{\cdot}585.$$

The above formulæ can be remembered easily in the following way. Remove the word *log* and write a line between each number and the base. The equations are then still true for the resulting fractions. Thus the last equation on p. 94 becomes:

$$\frac{y}{a} = \frac{y/10}{a/10}.$$

### The base *e*

There is one base that is commonly used as well as the base 10. This is the base $e$, which is nearly $2{\cdot}71828$. Logarithms to base $e$ are called "natural" logarithms or "Napierian" logarithms after Napier. The reason why the number $e$ is of importance in mathematics is that there are considerable simplifications in some of the more advanced work if numbers are expressed as powers of the base $e$ rather than of the base 10.

To four decimal places

$$\log_{10} e = 0{\cdot}4343\,;$$

to six significant figures

$$\log_{10} e = 0{\cdot}434294$$

and therefore $$\log_e 10 = \frac{1}{0{\cdot}434294} = 2{\cdot}3026.$$

Hence

$$\log_e a = \log_{10} a \div \log_{10} e = \log_{10} a \times \log_e 10 = 2{\cdot}3026 \times \log_{10} a.$$

In words, *to convert logarithms to base* 10 *into logarithms to base* e *multiply by* 2·3026.

*Example.*—Express 5 as a power of $e$.

If $5=e^x$, then $\log_{10} 5=x \log_{10} e$.

$\therefore\ x\times 0{\cdot}4343=0{\cdot}6990$

$\therefore\ x=\dfrac{0{\cdot}6990}{0{\cdot}4343}=1{\cdot}610.$ Hence $5=e^{1{\cdot}610}$.

In some books tables of logarithms to base $e$ are given. The log of a number to base $e$ can then be read directly from the tables provided the number lies within the range covered by tables. Most books give the logs of numbers from 1 to 5, and also $\log_e 10$, $\log_e 20$, etc. . . . $\log_e 100$. If a number does not lie between 1 and 5, express it as a product or a quotient of a number between 1 and 5 and one of the numbers 2, 10, 20, etc.

*Example.*—Find $\log_e 764{\cdot}8$ using logs to base 10 and from tables of logs to base $e$.

Using logs to base 10

$$\begin{aligned}\log_e 764{\cdot}8 &= 2{\cdot}3026 \log_{10} 764{\cdot}8\\ &= 2{\cdot}3026\times 2{\cdot}8836\\ &= 6{\cdot}6398\end{aligned}$$

Using logs to base $e$

$$\begin{aligned}\log_e 764{\cdot}8 &= \log_e (200\times 3{\cdot}824)\\ &= \log_e 2+\log_e 100+\log_e 3{\cdot}824\\ &= \log_e 2+2\log_e 10+\log_e 3{\cdot}824\\ &= 6{\cdot}6396.\end{aligned}$$

| *No.* | *Log* |
|---|---|
| 2 | 0·6931 |
| 100 | 4·6052 |
| 3·824 | 1·3413 |
| | 6·6396 |

*Example.*—The capacity of a cable which consists of a conducting cylindrical core of diameter $d$ surrounded by a coaxal conducting sheath of external diameter D, is $\dfrac{K}{2\log_e\left(\dfrac{D}{d}\right)}$ electrostatic units of capacity per cm. where K is the dielectric constant for the insulator between the core and the sheath. Given that 1 microfarad $=9\times 10^5$ electrostatic units and 1 in. $=2{\cdot}54$ cm.

show that the capacity of the cable is $\dfrac{0{\cdot}03883\text{K}}{\log_{10}\left(\dfrac{\text{D}}{d}\right)}$ microfarads per mile. Calculate this capacity when $d=0{\cdot}064$, $\text{D}=0{\cdot}3875$ and $\text{K}=3{\cdot}2$.

Capacity of 1 mile

$$=\text{capacity of } 5280\times 12\times 2{\cdot}54 \text{ cm.}$$

$$=\frac{5280\times 12\times 2{\cdot}54\text{K}}{2\log_e\left(\dfrac{\text{D}}{d}\right)} \text{ electrostatic units}$$

$$=\frac{5280\times 12\times 2{\cdot}54\text{K}}{2\times 9\times 10^5\log_e\left(\dfrac{\text{D}}{d}\right)} \text{ microfarads}$$

$$=\frac{5280\times 12\times 2{\cdot}54\text{K}}{2\times 9\times 10^5\times 2{\cdot}3026\log_{10}\dfrac{\text{D}}{d}} \text{ microfarads}$$

$$=\frac{0{\cdot}03883\text{K}}{\log_{10}\left(\dfrac{\text{D}}{d}\right)} \text{ microfarads.}$$

| *No.* | *Log* | |
|---|---|---|
| 5280 | 3·7226 | |
| 12 | 1·0792 | |
| 2·54 | 0·4048 | |
| | 5·2066 | 5·2066 |
| $18\times 10^5$ | 6·2553 | |
| 2·3026 | 0·3622 | |
| | 6·6175 | 6·6175 |
| 0·03883 | | $\bar{2}$·5891 |

When $\text{K}=3{\cdot}2$, $d=0{\cdot}064$, $\text{D}=0{\cdot}3875$,

$$\text{capacity}=\frac{0{\cdot}03883\times 3{\cdot}2}{\log\left(\dfrac{0{\cdot}3875}{0{\cdot}064}\right)}$$

$$=\frac{0{\cdot}03883\times 3{\cdot}2}{\log 0{\cdot}3875-\log 0{\cdot}064}$$

$$=\frac{0{\cdot}03883\times 3{\cdot}2}{0{\cdot}7821}$$

$$=0{\cdot}1588.$$

| *No.* | *Log* |
|---|---|
| 0·3875 | $\bar{1}$·5883 |
| 0·064 | $\bar{2}$·8062 |
| | 0·7821 |

| *No.* | *Log* |
|---|---|
| 0·03883 | $\bar{2}$·5891 |
| 3·2 | 0·5051 |
| | $\bar{1}$·0942 |
| 0·7821 | $\bar{1}$·8933 |
| | $\bar{1}$·2009 |

Hence the capacity is 0·159 microfarads per mile.

### Exercise XVI

Find the values of:

1. $\log_3 9$. 2. $\log_{20} 150$. 3. $\log_2 0{\cdot}1$. 4. $\log_4 \frac{1}{64}$.

5. Show that $\log a/\log b$ has the same value whatever base is used for the logarithms.

6. Find the values of $\log_e 3{\cdot}815$ and $\log_e 42{\cdot}6$ by using a table of Napierian logarithms. Also calculate their values by using a table of ordinary logarithms.

7. The self-inductance L henrys per mile of parallel conductors radius $r$ in., $d$ in. apart, is given by $L = 0{\cdot}000644\left\{\log_e \frac{d}{r} + \frac{1}{4}\right\}$. Calculate the self-inductance of 50 miles of wire if the diameter of the conductors is 0·5 cm., and they are 50 cm. apart.

8. If $n = \dfrac{\log (p_2/p_1)}{\log (v_2/v_1)}$, find $n$ if $p_1 = 100$, $p_2 = 15$, $v_1 = 4{\cdot}89$ and $v_2 = 24{\cdot}4$. Show that the value of $n$ is the same whatever base is used for the logarithms.

9. The change of entropy of a quantity of air during a compression is $C_v\left(\dfrac{\gamma - n}{n - 1}\right) \log_e \left(\dfrac{T_1}{T_2}\right)$. Calculate the value of this expression given that $C_v = 0{\cdot}169$, $\gamma = 1{\cdot}408$, $n = 1{\cdot}15$, $T_1 = 189{\cdot}3$ and $T_2 = 144$.

# CHAPTER V

## VARIATION, LINEAR LAWS

### Ratio. Homogeneous equations

In the equation $6x^2 + 4xy + 6y - 3 = 0$, the term $6x^2$ is of the second degree in $x$, the term $4xy$ is of the first degree in $x$ and the first degree in $y$, that is, of the second degree altogether, $6y$ is of the first degree and $-3$ is of no degree.

An equation like $x^2 - 3xy - 9y^2 = 0$ in which all the terms are of the same degree is called a *homogeneous* equation.

Although it is impossible to find the numerical values of two unknown quantities from one equation, *the ratio of*

*the two unknown quantities can be found from a homogeneous equation.*

*Example.*—Find the ratio of $y$ to $x$ if $2x+8y=16y-5x$.

Re-arranging the terms

$$16y-8y=2x+5x$$
$$\therefore\ 8y=7x.$$

Dividing by $8x$,

$$\frac{y}{x}=\frac{7}{8}.$$

*Example.*—Find the ratio of $p$ to $q$ if $p^2-2pq-4q^2=0$.

Dividing by $q^2$,

$$\frac{p^2}{q^2}-\frac{2p}{q}-4=0.$$

Writing $x$ for $\frac{p}{q}$,

$$x^2-2x-4=0$$
$$\therefore\ x=1\pm\sqrt{5}$$
$$\therefore\ \frac{p}{q}=1\pm\sqrt{5}.$$

## Proportion

If $a$, $b$, $c$ are proportional to $p$, $q$, $r$,

$$\frac{a}{p}=\frac{b}{q}=\frac{c}{r}.$$

In some problems involving proportion it is useful to use a single letter for each of the equal ratios.

*Example.*—The lengths of the sides $a$, $b$, $c$ of a triangle ABC are proportional to 7, 6 and 5. Find the value of $\frac{b^2+c^2-a^2}{2bc}$ [this ratio is cos A, see p. 308.]

We are given that $a:7=b:6=c:5$.

Write each ratio equal to $k$. Then,

$$\frac{a}{7}=\frac{b}{6}=\frac{c}{5}=k$$

$$\therefore\ a=7k,\ b=6k,\ c=5k.$$

$$\therefore\ \frac{b^2+c^2-a^2}{2bc}=\frac{36k^2+25k^2-49k^2}{2\,.\,6k\,.\,5k}$$

$$=\frac{12k^2}{60k^2}=\tfrac{1}{5}.$$

It should be noted that it is possible to find the numerical value of the ratio of two expressions only if all the terms in each of the expressions are of the same degree. For the example above, if the fraction had been $\dfrac{b^2-a}{2c}$, we should get

$$\frac{b^2-a}{2c}=\frac{36k^2-7k}{10k}=\frac{36k-7}{10}.$$

As the fraction contains $k$, which is unknown, it is impossible to find its numerical value.

### Variation

The expression "$y$ varies as $x$", which is written $y \propto x$, means $y=kx$, where $k$ is a fixed number, for every value of $x$. To distinguish this kind of variation from others we say that $y$ varies directly as $x$.

Other simple laws are obtained when $y$ varies as some other expression containing $x$ such as $x^2$, $1/x$, $\log x$. If $y \propto x^2$, then $y=kx^2$ is the law connecting $y$ and $x$.

$y \propto 1/x$ is given the special name of "inverse variation"; we say "$y$ varies inversely as $x$." Thus $y$ varies inversely as $\sqrt{x}$ means $y \propto 1/\sqrt{x}$, or $y=k/\sqrt{x}$.

*Example.*—Assuming that at a constant temperature the pressure, $p$ lb. wt. per sq. ft., of a gas varies inversely as its volume, $v$ cu. ft. (Boyle's Law), find the law connecting $p$ and $v$ if $p=20$ when $v=4$. Also find $p$ when $v=15$, and $v$ when $p=30$.

Since $p$ varies inversely as $v$, $p=k/v$.
But $p=20$ when $v=4$.
$\therefore\ 20=k/4 \qquad \therefore\ k=80.$
$\therefore\ p=80/v.$
When $v=15$, $p=80/15=16/3=5\frac{1}{3}$.
When $p=30$, $80/v=30 \qquad \therefore\ v=80/30=2\frac{2}{3}$.

## Graphs of variation

It is important to know the shapes of the graphs of different kinds of variation. The graph of direct variation $y=kx$ is a straight line of gradient $k$ through $(0, 0)$.

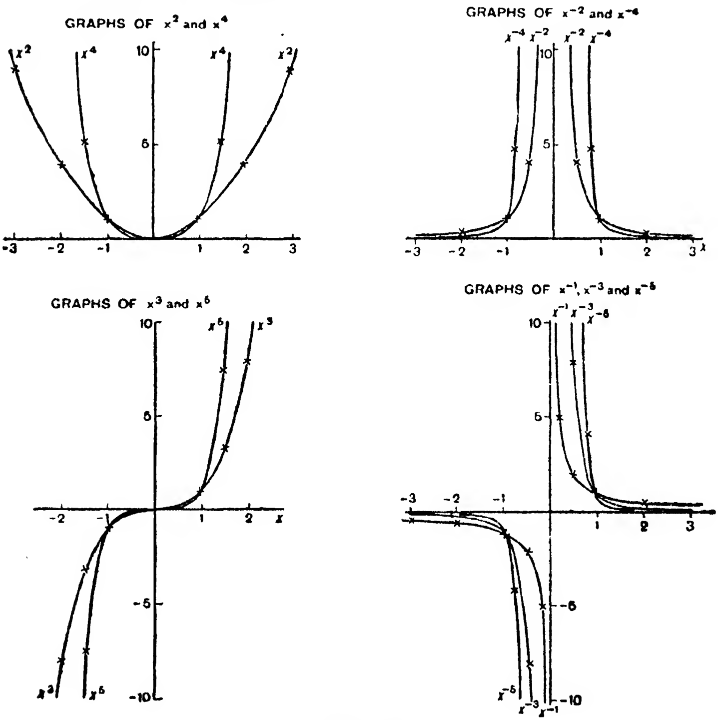



FIG. 29.

If $y \propto x^n$, $y=kx^n$ and the graph of this equation is of the same shape as the graph of $y=x^n$, because it is obtained by multiplying each ordinate of $y=x^n$ by $k$. Hence it is sufficient if we know the shape of the graph of $y=x^n$ for different values of $n$.

At the top of Fig. 29 the graphs of $y=x^2$ and $y=x^4$, and the graphs of $y=\frac{1}{x^2}$ and $y=\frac{1}{x^4}$ are shown. Since in all these graphs $y$ has the same value if the sign of $x$ is changed, the graphs are symmetrical about the vertical axis.

At the bottom of Fig. 29 the graphs of $y=x^3$ and $y=x^5$, and the graphs of $y=\frac{1}{x}$, $y=\frac{1}{x^3}$, $y=\frac{1}{x^5}$ are shown. In each of these graphs, if the sign of $x$ is changed $y$ keeps the same numerical value, but changes sign. For instance, if P and $Q$ are the points on $y=x^3$ given by $x=+a$ and $x=-a$, as in Fig. 30, PQ is bisected at O. When this is so a graph is said to be symmetrical about the origin O.

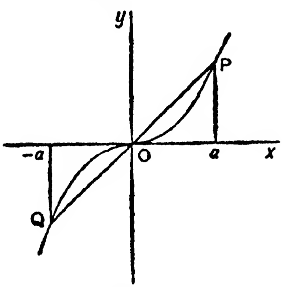


Fig. 30.

Since $x^{2\cdot3}$ lies between $x^2$ and $x^3$ for every positive value of $x$, the graph of $y=x^{2\cdot3}$ for positive values of $x$ lies between the graphs of $y=x^2$ and $y=x^3$. Between $x=0$ and 1, $y=x^2$ comes above $y=x^3$ and $y=x^{2\cdot3}$ lies between them. When $x>1$, $y=x^2$ is below $y=x^3$ and $y=x^{2\cdot3}$ is between them again. The three graphs are shown in Fig. 31.

When we expect that values of $y$ and $x$ obtained from an experiment are such that $y$ varies as some power of $x$, by plotting $y$ against $x$ we can see if the graph looks like one of the forms of the graph of $x^n$ in Fig. 29, but the fact that the shapes just look similar is not a proof that $y \propto x^n$. We shall see on p. 115 how we can test whether a relationship of this type really exists.

GRAPH OF $x^{2\cdot3}$

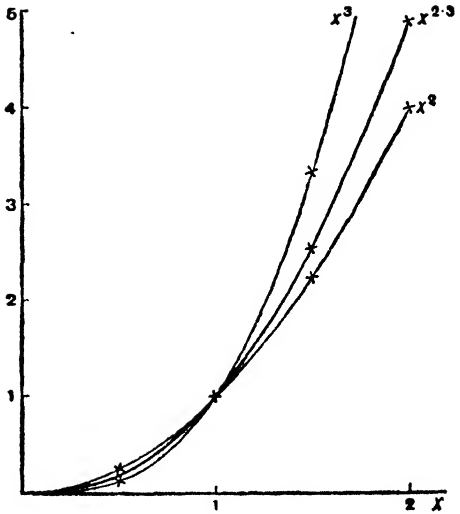



FIG. 31.

**Joint Variation**

If the volume of a cylinder of radius $r$ ft. and height $h$ ft. is V cu. ft., $V=\pi r^2 h$,

$$\therefore\ h=\frac{V}{\pi r^2}.$$

Now, if $r$ is kept fixed and V varied, $(1/\pi r^2)$ is constant, and hence $h \propto V$. Fig. 32 shows three cylinders of the same radius but different volumes ; in this figure $h \propto V$.

If, however, V is kept fixed and $r$ varied, $\dfrac{V}{\pi}$ is constant and $h \propto \dfrac{1}{r^2}$. Fig. 33 shows three cylinders of the same volume but with different radii ; in this figure $h \propto \dfrac{1}{r^2}$.

Thus we may say that $h$ varies directly as V and inversely as the square of $r$. This is called "joint variation," and we may say that $h$ varies jointly as V and $1/r^2$.

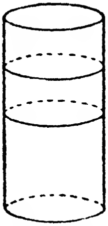


FIG. 32.

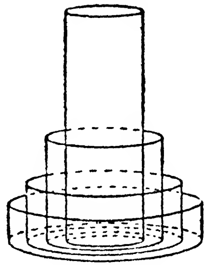


FIG. 33.

In general, if the value of $z$ depends on $x$ and $y$, and if $z \propto x$ when $y$ is constant and $z \propto y$ when $x$ is constant, then $z \propto xy$ when $x$ and $y$ both vary, which means that the law relating $z$, $x$ and $y$ is $z = kxy$, where $k$ is a constant.

*Example.*—When a given beam is placed on two supports A and B, $l$ in. apart, and a load W lb. wt. is hung from the middle point of the beam, the deflection $y$ in. produced by the load varies jointly as W and $l^3$.

In an experiment it is found that $y = 0{\cdot}35$ when $W = 5$ and $l = 40$. Find the relation between $y$, W and $l$, and from it calculate the deflection produced by a load 8 lb. wt. when $l = 60$.

Since $y \propto W$, and $y \propto l^3$,

$$\therefore\ y \propto Wl^3,$$

that is $y = kWl^3$, where $k$ is constant.

Substituting $y = 0{\cdot}35$, $W = 5$, $l = 40$

$$0{\cdot}35 = k \times 5 \times 40^3$$

$$\therefore\ k = \frac{0{\cdot}35}{5 \times 40^3} = \frac{0{\cdot}07}{40^3} = 1{\cdot}094 \times 10^{-6}$$

$$\therefore\ y = 1{\cdot}094 \times 10^{-6} Wl^3.$$

When $W=8$ and $l=60$,

$$y=1{\cdot}094\times10^{-6}\times8\times60^3=1{\cdot}89.$$

Therefore the deflection is 1·89 in.

*Example.*—It is known that the electrical resistance, R ohms, of a wire of given material varies directly as its length and inversely as the area of its cross section. Find the relation between the resistance of a wire, its length $l$ cm. and its weight W gm. wt.

Let the area of the cross-section be A sq. cm. Then $R\propto l$ and $R\propto\frac{1}{A}$.

$$\therefore\ R=\frac{kl}{A}.$$

But the weight of the wire varies as its volume $lA$ cu. cm.

$$\therefore\ W=k_1lA$$

$$\text{whence } A=\frac{W}{k_1l}.$$

Substituting this value of A in the value of R.

$$R=kl\Big/\left(\frac{W}{k_1l}\right)=\frac{kk_1l^2}{W}.$$

$$\therefore\ R=\frac{Kl^2}{W},$$

where K is a constant for a wire of given material.

## Exercise XVII

1. Find the ratio of $r_1$ to $r_2$ if $l_1/r_1=l/(r_2-r_1)$.

2. Find the ratio of $p$ to $q$ if $3(p+2q)=5(p-q)$.

3. Find the ratio of $m$ to $n$ if $(m^2-n^2)/(m^2+n^2)=2/3$.

4. If six equal circles are drawn so that each touches two others and they all touch two concentric circles of radii R and $r$, show that $(R-r)/(R+r)=\sin 30°$. Hence find the ratio of R to $r$.

**5.** If in the figure in Question 4, there are $n$ circles touching the two concentric circles, show that $(R-r)/(R+r)=\sin\dfrac{\pi}{n}$, and find the ratio of R to $r$.

**6.** If a circle of radius $b$ and centre C rolls on a fixed circle of centre A and radius $a$, the angular velocity $\omega$ of the rolling circle and $\Omega$ the angular velocity of AC are related by the equation $a\Omega=b(\omega-\Omega)$. Find the ratio of $\Omega$ to $\omega$.

**7.** If $\dfrac{R_1}{R_2}=\sqrt{\dfrac{R_1{}^2+R_3{}^2}{R_2{}^2+R_4{}^2}}$ prove that $R_1 : R_2 = R_3 : R_4$.

**8.** The acceleration $f$ of two masses $m_1$ and $m_2$ connected by a string over a smoothly-running pulley is given by $f=\left(\dfrac{m_1-m_2}{m_1+m_2}\right)g$. Find the ratio of $m_1$ to $m_2$ in terms of $f$ and $g$.

**9.** If there is no current in the galvanometer arm of a Wheatstone bridge the resistances of the other four wires and the currents in them are related by the equations $R_1i_1=R_4i_2$, $R_2i_1=R_3i_2$. Prove that $R_1 : R_2 = R_4 : R_3$.

**10.** If $l$, $m$ and $n$ are proportional to 3, 4 and 6, calculate the values of

(i) $\dfrac{l+m+n}{l+m-n}$, (ii) $\dfrac{lm-n^2}{mn-l^2}$, (iii) $\dfrac{n^4}{l^2m^2}$.

**11.** If $a$, $b$, $c$ are proportional to $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ find the values of

(i) $\dfrac{a+b}{b+c}$, (ii) $\dfrac{c^2-a^2}{c^2-b^2}$,

and show that it is impossible to find the values of $\dfrac{c^2}{a}$ and $\dfrac{a+c}{ab-c^2}$.

**12.** The air resistance R lb. wt. to the motion of a certain locomotive and 12 coaches varies as the square of the velocity V m.p.h. If the resistance is 710 lb. wt. when the speed of the train is 30 m.p.h., find R in terms of V. Draw a graph to show the values of R from V = 0 to 120. From the graph read off the values of R when V = 48, 86 and 112, and the value of V when R = 8000.

**13.** If water flows out of a tank through a tap in the bottom the volume V cu. ft. which flows out per minute varies as the square root of the depth $x$ ft. of water left in the tank. If V = 6·5 when $x=6$, find the relation between V and $x$. How fast is the water flowing out when it is 2 ft. deep? Draw a graph of V against

$x$ from $x=0$ to 6. From it read off the value of V when $x=4{\cdot}7$ and the value of $x$ when $V=4{\cdot}2$.

14. If $h=\frac{glr}{v^2}$ in what ratio is $h$ decreased if $r$ is halved and $v$ is doubled ?

15. If $y$ varies inversely as $\sqrt{x}$ in what ratio is $y$ changed if $x$ is multiplied by $n$ ?

16. For a given voltage the current in a wire varies inversely as its resistance. If the current is 10 ampères for a resistance of 2·5 ohms find the relationship between the current and resistance, and draw a graph of this relation for resistances from 0·5 to 5 ohms.

17. In experiments on the wave resistance of ships it is found that in order to produce similar behaviour in a model and a ship the ratio (length)/(velocity)$^2$ must be the same for the model and the ship. Express the velocity $v_m$ of the model in terms of the velocity $v$ of the ship and the lengths $l$ and $l_m$ of the ship and model respectively.

18. If $z$ varies jointly as $x$ and $y$ state the formula relating $x$, $y$ and $z$ with a constant of variation. Find this constant if $z=10$ when $x=5$ and $y=4$. Find $z$ when $x=0{\cdot}2$, $y=40$, and find $x$ when $z=16$, $y=15$.

19. $x$ varies jointly as $f$ and as the square of $t$. Express this as a formula. If $x=64$ when $f=32$ and $t=4$ find the constant of variation. Find $x$ when $f=3$ and $t=10$, and find $t$ when $x=100$, $f=2$. Also make $t$ the subject of the formula.

20. The volume V of a cone varies as its height $h$ and as the square of its base radius $r$. It is found experimentally that $V \simeq 100$ cu. in. when $h=6$ in., $r=4$ in. Find the formula connecting V, $h$ and $r$. If the height of the cone is doubled and the base radius is trebled in what ratio is the volume of the cone increased ?

21. The pressure of a gas $p$ lb./ft.$^2$ varies directly as its absolute temperature T° when its volume is constant, and inversely as its volume $v$ cu. ft. when its temperature is constant. Express this by means of a formula with a constant of variation $k$. Find $k$ for 1 lb. hydrogen if $T=303$ when $p=20{,}000$ and $v=35$. From the formula find T when $p=10{,}000$ and $v=5$, and find $v$ when $T=100$ and $p=5000$.

22. The resistance R ohms of a wire of given material varies directly as its length $l$ cm. when its diameter $d$ cm. is constant, and inversely as the square of its diameter when its length is constant.

Write down the formula connecting R, $l$ and $d$. Find the constant of variation if $R=2\cdot5$ for a wire 100 cm. long, 0·015 cm. diameter. From the formula (*a*) calculate R for a wire 150 cm. long, 0·02 cm. diameter, (*b*) find what length of wire of 0·1 cm. diameter will have a resistance of 3·4 ohms.

Express each of the following statements as a formula with a constant of variation $k$.

**23.** The moment of inertia I of a cone about its axis varies as its height $h$ and as the fourth power of its base radius $r$.

**24.** The torque N lb. wt. ft. required to twist one end of a wire of radius $r$ in. and length $l$ in. through $\theta$ radians when the other end is fixed, varies directly as $l$ and as $\theta$ and varies as the fourth power of $r$.

**25.** The bending moment M at the middle point of a beam supported at its ends varies directly as its weight W and as its length $l$.

**26.** The exposure E necessary for a photograph varies as the square of the stop $f$ used and inversely as the speed $s$ of the plate.

**27.** The self-inductance L of a solenoid varies as the square of the number of turns $n$, as the area of the cross-section A and inversely as the length $l$.

**28.** The period T of small horizontal oscillations of a body hung by a vertical wire varies as the square root of the length $l$ of the wire and inversely as the square of the diameter $d$ of the wire.

**29.** The capacitance C farads of a parallel plate condenser of $n$ plates varies directly as $(n-1)$ and as A sq. cm. the area of a plate, and inversely as $d$ cm. the distance between the plates. If the capacitance is 424 micromicrofarads for a condenser with 9 plates each of 9 sq. cm. and 1 mm. apart, find the formula connecting C, $n$, A and $d$, and from it calculate the capacitance of a condenser with 11 plates each of 50 sq. cm. with 2 mm. between the plates.

**30.** The force F required to stretch a wire of given material varies directly as the area of its cross-section A, directly as the distance $x$ by which the wire is extended and inversely as $l$ the natural length of the wire. It is found that a force of 50 lb. wt. is required to stretch a wire 12 ft. long, 0·02 in. radius through a distance 0·15 in. Find the law connecting F, A, $x$ and $l$. From

it calculate the force required to stretch a wire 6 ft. long, 0·03 in. radius, made of the same material, through a distance 0·05 in.

**31.** For ships of the same shape the wave resistance varies as $l^{2+k}\,v^{2-2k}$ where $l$ is a linear dimension of the ship (e.g. its length) and $v$ is its velocity. Show that if the linear dimensions of a ship are $n$ times those of a model and the velocity of the ship is $\sqrt{n}$ times the velocity of the model, then the resistance to the motion of the ship is $n^3$ times the resistance to the model.

**32.** The resistance R of a wire varies as its length $l$ and inversely as the square of its diameter $d$. The weight W of the wire varies as its length and as the square of its diameter. Find the formula connecting R, W and $l$. The resistance of a bronze wire used for a telephone line is 22 ohms per mile and the weight of the wire is 80 lb. per mile; find the resistance of a line of 8 miles using wire of the same material, but weighing 120 lb. per mile.

**33.** A circuit has a variable condenser and a fixed induction coil. The resonance frequency $f$ of the circuit varies inversely as the square root of the capacity C of the condenser. The capacity of the condenser varies inversely as the distance $d$ between its plates. How does the resonance frequency depend on $d$?

**34.** If lamps of given candle-power are constructed to work at a definite efficiency for all values of the voltage V, the length of the filament $l$, its diameter $d$, its resistance R and the current $i$ through it are related to one another and to V in the following way: $i \propto \frac{1}{V}$, $d^3 \propto i^2$, $d \propto 1/l$, $R \propto l/d^2$. Prove that $R \propto V^2$, $d \propto V^{-2/3}$ and $l \propto V^{2/3}$.

Sketch roughly, both for positive and negative values of the variables, the graphs of the following equations.

**35.** $y = 2x^3$. **36.** $y = -2x^3$. **37.** $y = 4/x^2$. **38.** $S = 16t^2$.
**39.** $pv = 50$. **40.** $y^2 = 4x$. **41.** $v^2 = 20h$. **42.** $Rd^2 = 100$.

Indicate by a rough sketch the graph of $y$ against $x$ when:

**43.** $y \propto x^2$. **44.** $y \propto \frac{1}{x}$. **45.** $y \propto x^{1\cdot5}$.
**46.** $y \propto x^{0\cdot7}$. **47.** $y \propto x^{-1\cdot2}$.

### Laws containing two constants

The laws or equations of variation which we have used so far have each contained only one constant and in each case

that constant has been found by substituting some special pair of values of the variable quantities in the epuation.

In many cases physical and mechanical quantities are related by more complicated laws containing two or more constants. The simplest law involving two constants is the equation $y=ax+b$ where $x$ and $y$ are the variables and $a$ and $b$ the constants. $a$ and $b$ can be found from the pair of simultaneous equations obtained by substituting in this equation two pairs of values of $x$ and $y$. Because the graph of $y=ax+b$ is a straight line of gradient $a$ (pp. 35 and 36) this law is sometimes called a " straight line law " or " linear law." It includes the equation of direct variation $y=kx$ as a special case in which the graph passes through the origin.

Other simple laws with two constants are obtained by replacing $x$ by $x^2$, $\frac{1}{x}$ or some other expression in the equation $y=ax+b$. For instance, the law $y=ax^3+b$ is obtained by replacing $x$ by $x^3$. The graph of this law is not of course a straight line if we plot $y$ against $x$, but it is a straight line if we plot $y$ against $x^3$. The constants in any two-constant law are found by substituting two pairs of values of $x$ and $y$ in the equation.

*Example.*—The resistance to the motion of a motor car is R lb. wt. at a speed of $v$ m.p.h. and R is related to $v$ by the equation $R=av^2+b$. If $R=53$ when $v=10$, and $R=65$ when $v=20$ find the law connecting R and $v$, and also find R when $v=70$.

Substituting the given values of R and $v$,

$$53=100a+b$$

$$65=400a+b$$

Subtracting $\qquad 12=300a \qquad \therefore a=0{\cdot}04.$

Substituting in the first equation

$$53=100\times 0{\cdot}04+b$$

$$\therefore b=53-4=49.$$

Hence the required law is

$$R = 0{\cdot}04v^2 + 49.$$

When $v = 70$,

$$R = (0{\cdot}04 \times 4900) + 49 = 245.$$

Another common type of law containing two constants occurs when one variable quantity $y$ varies as some unknown power of another variable quantity $x$. Suppose $y \propto x^n$, then $y = kx^n$, where both the constants $k$ and $n$ are unknown.

To find these constants we have to take logarithms. The method used is shown in the following example.

*Example.*—During the expansion of a gas without loss of heat the pressure and volume are related by the equation $pv^n = c$, where $n$ and $c$ are constants for a particular gas. If $p = 50$ when $v = 20$ and $p = 280$ when $v = 5{\cdot}14$ find the values of $n$ and $c$, and state the relation between $p$ and $v$.

Since

$$pv^n = c$$
$$\log (pv^n) = \log c$$
$$\therefore \log p + n \log v = \log c.$$

Substituting the values of $p$ and $v$,

$$\log 50 + n \log 20 = \log c$$
$$\log 280 + n \log 5{\cdot}14 = \log c.$$

Hence,

$$1{\cdot}6990 + 1{\cdot}3010n = \log c$$
$$2{\cdot}4472 + 0{\cdot}7110n = \log c.$$

Subtracting,

$$-0{\cdot}7482 + 0{\cdot}5900n = 0$$
$$\therefore n = \frac{0{\cdot}7484}{0{\cdot}5900} \simeq 1{\cdot}268.$$

Substituting,

$$1{\cdot}6990 + 1{\cdot}3010 \times 1{\cdot}268 = \log c$$
$$\therefore \log c = 1{\cdot}6990 + 1{\cdot}649$$
$$= 3{\cdot}348$$
$$\therefore c \simeq 2228.$$

| *No.* | *Log* |
|---|---|
| 1·3010 | 0·1142 |
| 1·268 | 0·1031 |
| 1·649 | 0·2173 |

Hence the law connecting $p$ and $v$ is approximately

$$pv^{1{\cdot}27} = 2230.$$

**Straight line law from experimental values**

If it is thought that the values of $x$ and $y$ obtained from an experiment are related by a straight line law $y = ax + b$, this conjecture can be tested by plotting the values of $y$ against the corresponding values of $x$. If the points lie nearly on a straight line then we are justified in concluding that the relationship between $x$ and $y$ is represented very nearly by the equation $y = ax + b$. The graph of this equation is taken to be a straight line drawn as evenly as possible between the points, and the constants $a$ and $b$ are found by substituting the values of $x$ and $y$ at two points *on the line*, which are generally not points given by the experimental values. Alternatively $a$ can be found from the gradient of the line, and $b$, being the value of $y$ at $x = 0$, can be read directly from the graph *provided that the line $x = 0$ comes on the graph paper.*

*Example.*—In an engine test the values of the indicated horse power I, brake horse power B and steam consumption S lb. per hour were found to be :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| I | .. | 1·197 | 4·17 | 5·85 | 7·96 | 8·91 | 10·77 | 14·68 |
| B | .. | 0 | 2·91 | 4·55 | 6·32 | 7·45 | 9·05 | 12·3 |
| S | .. | 51·6 | 171·8 | 216 | 316 | 351 | 410 | 564 |

Show that the relations between B and I and between S and I are both approximately linear laws. Find approximations to each of these laws.

The graphs of B against I and S against I are plotted from the above table in Fig. 34 and Fig. 35. The points in each figure lie nearly on a straight line, showing that in both figures a linear law is a very good approximation to the relation between the variables. In Fig. 34 let $B = a + bI$. At the point C, $I = 2$, $B = 1$ and at D, $I = 12$, $B = 10$.

$$\therefore\ 10 = a + 12b,$$
$$1 = a + 2b,$$

whence $9 = 10b$. $\quad \therefore\ b = 0{\cdot}9$ and $a = 1 - 1{\cdot}8 = -0{\cdot}8$.

Hence $\quad B = -0{\cdot}8 + 0{\cdot}9I$.

GRAPH OF BRAKE HORSE POWER B AGAINST INDICATED HORSE POWER I

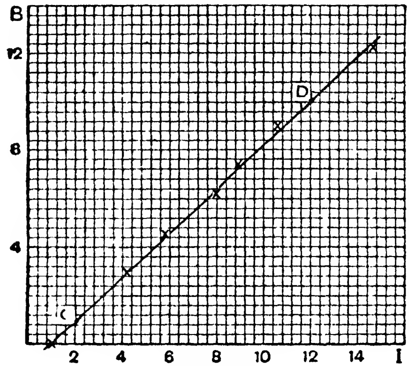



FIG. 34.

In Fig. 35 let $S = a + bI$. The points E, $I = 2$, $S = 80$, and F, $I = 12$, $S = 460$, lie on the line.

$$\therefore\ 460 = a + 12b,$$
$$80 = a + 2b,$$

whence $380 = 10b$. $\quad \therefore\ b = 38,\ a = 80 - 76 = 4.$

Hence $S = 4 + 38I$.

GRAPH OF STEAM CONSUMPTION S LB/HR AGAINST INDICATED HORSE POWER I

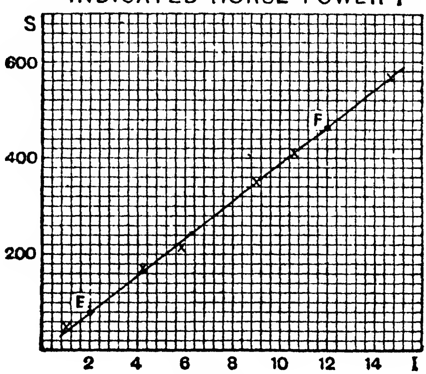



FIG. 35.

**Laws which can be converted to straight line laws**

The equation $xy = ax^2 + b$ becomes $Y = aX + b$ if we write $xy = Y$ and $x^2 = X$. The graph of Y against X is then a straight line and this means that, if *we plot $xy$ against $x^2$, the graph of $xy = ax^2 + b$ is a straight line.* It is, of course, a curve when $y$ is plotted against $x$.

It is therefore possible to test whether $x$ and $y$ are related by a law of the form $xy = ax^2 + b$ by plotting $xy$ against $x^2$ and seeing whether the points lie on a straight line or not. If they do then the approximate values of $a$ and $b$ can be found in the same way as before, by drawing the straight line which lies most evenly between the points and calculating $a$ and $b$ by substituting the co-ordinates of two points in the equation remembering that these co-ordinates are values of $xy$ and $x^2$, not of $y$ and $x$.

In the same way any two-constant law which can be put in the form $Y = aX + b$ where X and Y stand for expressions containing $x$ and $y$ gives a straight line graph if we plot Y against X.

*Example.*—In the following table $w$ watts is the iron loss in a dynamo due to hysteresis when $f$ is the frequency of the current. Show that the relationship between $w$ and $f$ is of the form $w = af + bf^2$ and find the approximate values of $a$ and $b$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $w$ | .. | 45 | 62 | 76 | 90 | 120 |
| $f$ | .. | 23 | 30 | 35·5 | 40 | 49·7 |

Dividing by $f$ the given equation becomes

$$\frac{w}{f} = a + bf$$

which has a straight line graph if $w/f$ is plotted against $f$. The values of $w/f$ are tabulated below and the points in Fig. 36 are plotted from this table.

| | | | | | | |
|---|---|---|---|---|---|---|
| $f$ | .. | 23 | 30 | 35·5 | 40 | 49·7 |
| $\frac{w}{f}$ | .. | 1·957 | 2·067 | 2·141 | 2·230 | 2·414 |

These points lie nearly on the straight line AB, thus showing that the assumed law is the relation between $\frac{w}{f}$ and $f$. The point A is given by $f=20$, $\frac{w}{f}=1{\cdot}905$; and B by $f=50$, $\frac{w}{f}=2{\cdot}39$.

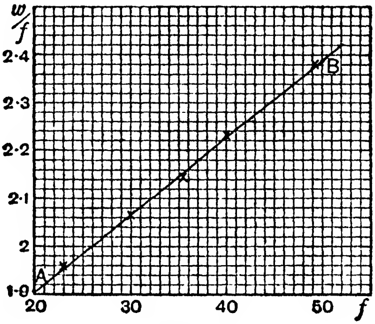



FIG. 36.

Substituting these values,

$$2{\cdot}39 = a + 50b,$$
$$1{\cdot}905 = a + 20b,$$
$$\therefore\ 0{\cdot}485 = 30b$$
$$\therefore\ b = 0{\cdot}0162, \text{ and } a = 1{\cdot}905 - 0{\cdot}324 = 1{\cdot}581.$$

Hence the law relating $w$ and $f$ is :

$$\frac{w}{f} = 1{\cdot}581 + 0{\cdot}0162f, \text{ or } w = 1{\cdot}581f + 0{\cdot}0162f^2.$$

### $y$ varying as a power of $x$

If $y \propto x^n$, $y = kx^n$ and hence taking logarithms as on p. 111,

$$\log y = \log k + n \log x.$$

If in this equation we write X for $\log x$, Y for $\log y$ and $a$ for $\log k$, we get $Y = a + nX$ which has a straight line graph if Y is plotted against X, that is $\log y$ plotted against $\log x$.

Conversely, if log $y$ plotted against log $x$ gives a straight line graph, $y$ varies as some power of $x$ and the actual power $n$ and the constant of variation $k$ can be found by substituting the values of log $y$ and log $x$ at two points on the line in the equation $\log y = \log k + n \log x$.

*Example.*—The following table gives the quantity of water Q cu. ft. per sec. which flows through a V-shaped notch when the depth of the water is H ft. Show that Q varies as a power of H and find approximately the law connecting Q and H.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| H .. | 0·166 | 0·209 | 0·270 | 0·338 | 0·392 | 0·473 | 0·550 |
| Q .. | 0·0293 | 0·0509 | 0·0987 | 0·171 | 0·258 | 0·410 | 0·602 |

If $Q = kH^n$, $\log Q = \log k + n \log H$. Hence we tabulate the values of log Q and log H.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| log H .. | $\bar{1}$·220 | $\bar{1}$·320 | $\bar{1}$·431 | $\bar{1}$·529 | $\bar{1}$·593 | $\bar{1}$·675 | $\bar{1}$·740 |
| log Q .. | $\bar{2}$·467 | $\bar{2}$·707 | $\bar{2}$·994 | $\bar{1}$·233 | $\bar{1}$·412 | $\bar{1}$·613 | $\bar{1}$·780 |

Points are plotted from this table in Fig. 37. Remember that $\bar{2}$·467 means $-2 + 0{\cdot}467$ so that if the origin is taken at

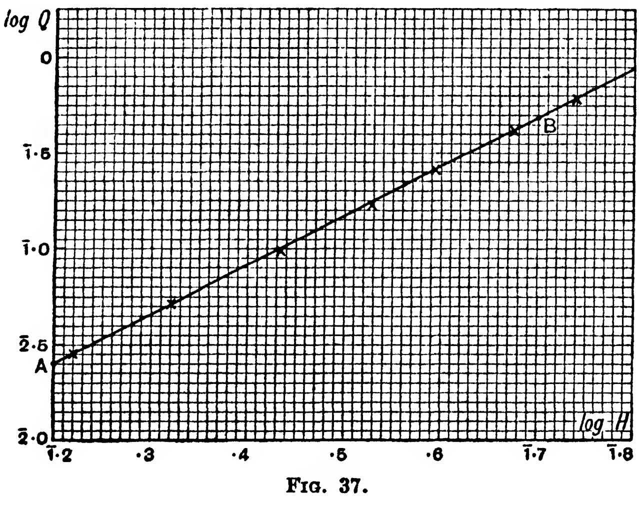



Fig. 37.

$-2$ on the vertical axis $\bar{2}{\cdot}467$ is $0{\cdot}467$ above it; $\bar{1}$ is 1 above the origin because $\bar{1} = -1 = -2+1$.

Since the points lie very nearly on a straight line, Q does vary as a power of H. At the point A on the line,

$$\log H = \bar{1}{\cdot}2 = -0{\cdot}8, \ \log Q = \bar{2}{\cdot}40 = -1{\cdot}60,$$

and at the point B,

$$\log H = \bar{1}{\cdot}7 = -0{\cdot}3, \ \log Q = \bar{1}{\cdot}69 = -0{\cdot}31.$$

Substituting these values in the equation

$$\log Q = \log k + n \log H,$$

we get

$$-1{\cdot}60 = \log k - 0{\cdot}8n,$$
$$-0{\cdot}31 = \log k - 0{\cdot}3n.$$

$$\therefore \ 1{\cdot}29 = 0{\cdot}5n \text{ and } n = \frac{1{\cdot}29}{0{\cdot}5} = 2{\cdot}58$$

$$\therefore \ \log k = -0{\cdot}31 + 0{\cdot}3 \times 2{\cdot}58 = -0{\cdot}31 + 0{\cdot}774 = 0{\cdot}463$$

$$\therefore \ k \simeq 2{\cdot}9.$$

Hence the law connecting Q and H is $Q = 2{\cdot}9\, H^{2{\cdot}58}$.

## Exercise XVIII

**1.** If $y = axy + bx^2$ and $y = 25$ when $x = 2$, $y = 50$ when $x = 6$, find $a$ and $b$. Also calculate the value of $y$ when $x = 10$.

**2.** If $u$ cm. and $v$ cm. are the distances of an object and its image from a lens, $\frac{1}{u} = \frac{a}{v} + b$, where $a$ and $b$ are constants. In an experiment it was found that $v = 21{\cdot}45$ when $u = 300$ and $v = 24{\cdot}25$ when $u = 100$. Find the law connecting $u$ and $v$. Find also the focal length, $f$ cm., of the lens which is given by $f = 1/b$.

**3.** If a body moves $s$ ft. in $t$ sec. with constant acceleration, $s = ut + \frac{1}{2}ft^2$, where $u$ and $f$ are constant. If $s = 600$ when $t = 24$ and $s = 3000$ when $t = 60$, find $u$ and $f$.

**4.** If $y = ax^n$ and $y = 35{\cdot}2$ when $x = 290$, $y = 77{\cdot}3$ when $x = 370$, find the values of $a$ and $n$. Also calculate the value of $y$ when $x = 410$.

**5.** If T sec. is the period of oscillation of a weight W lb., which is hanging by a vertical spring, T varies as $W^n$. It is found in an

experiment that $T=0{\cdot}471$ when $W=2$ and $T=0{\cdot}745$ when $W=5$. Find the law connecting T and W. What must the weight be for the period of oscillation to be 1 sec.?

6. When a body cools the temperature, $\theta°$ C., of the body above the surrounding air is related to the time $t$ min. that the body has been cooling by the equation $\log \theta = a + bt$. In an experiment $\theta = 27$ when $t = 4$ and $\theta = 5{\cdot}3$ when $t = 20$. Find $a$ and $b$ and hence show that $\theta = 40{\cdot}6 \times 10^{-0{\cdot}0442t}$.

Verify that the given variables are related by a law of the given form. Find the approximate value of the constants in the law. Write out the law and check it for one value of the independent variable.

7. R ohms is the resistance of a conductor at $t°$ C. Law: $R = R_0(1 + at)$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ .. | 2 | 4 | 6 | 8 | 10 | 12 |
| R .. | 100·9 | 101·5 | 102·5 | 103·1 | 104 | 104·8 |

Also express R in the form $R = k\{1 + b(t-2)\}$.

8. $p$ atmospheres is the osmotic pressure of a solution at $t°$ C. Law: $p = at + b$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $p$ .. | 24·8 | 25·3 | 25·7 | 26·2 | 26·6 | 27·0 |
| $t$ .. | 0 | 5 | 10 | 15 | 20 | 25 |

9. W lb. wt. is the weight of one hundred $\frac{1}{4}$-in. Whitworth hexagon screws of length $l$ in. Law: $W = al + b$.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $l$ .. | $\frac{1}{4}$ | $\frac{5}{16}$ | $\frac{3}{8}$ | $\frac{7}{16}$ | $\frac{1}{2}$ | $\frac{5}{8}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | 1 | $1\frac{1}{8}$ | $1\frac{1}{4}$ |
| W .. | 1·75 | 1·82 | 1·89 | 1·95 | 2·02 | 2·15 | 2·29 | 2·42 | 2·56 | 2·69 | 2·83 |

10. B tons wt. is the breaking load of No. 4/37 wire ropes of diameter $d$ in. Law: $B = ad^2 + b$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $d$ .. | $\frac{3}{8}$ | $\frac{1}{2}$ | $\frac{5}{8}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | 1 | $1\frac{1}{8}$ | $1\frac{1}{4}$ | $1\frac{3}{8}$ |
| B .. | 3·7 | 7·9 | 11·8 | 16·6 | 22·1 | 28·5 | 35·6 | 43·6 | 55·1 |

11. $l$ in. is the gauge length between two marks on a mild steel rod before a breaking test and $e$ is the percentage elongation of the length $l$ when the rod breaks. Law: $e = \frac{a}{l} + b$. [Plot $e$ against $\frac{1}{l}$.]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $l$ .. | 2 | 4 | 6 | 8 | 10 | 12 | 14 |
| $e$ .. | 64·2 | 41·6 | 34·3 | 30·4 | 28·2 | 26 8 | 25·6 |

12. W lb. wt. is the load applied to a bar at a distance $x$ in. from a fulcrum in an experiment on moments. Law: $W=\frac{a}{x}+b$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ .. .. | 8 | 10 | 12 | 14 | 16 | 18 |
| W .. .. | 48 | 38·2 | 30·8 | 26·1 | 22 | 19·4 |

13. $l$ cm. is the length of the simple pendulum which has the same period of oscillation as a rigid body oscillating about a horizontal axis $h$ cm. from its centre of gravity. Law: $l=ah+\frac{b}{h}$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $h$ .. | 5 | 7·5 | 10 | 12·5 | 15 | 20 | 25 |
| $l$ .. | 50·6 | 37·5 | 32·4 | 30·3 | 29·9 | 31·3 | 34·0 |

14. H is the horse-power absorbed in drilling cast iron with a drill of diameter $d$ in. Law: $H=a\sqrt{d}+\frac{b}{\sqrt{d}}$. [Plot $H\sqrt{d}$ against $d$.]

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $d$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 |
| H .. | 1·58 | 1·34 | 1·27 | 1·26 | 1·29 | 1·34 | 1·39 | 1·46 |

15. $f$ lb./in.$^2$ is the buckling stress of a strut in which the ratio of length to radius is $x$. Law: $f=\frac{a}{b+x^2}$. [Plot $\frac{1}{f}$ against $x^2$.]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ .. | 10 | 20 | 40 | 60 | 80 | 100 | 120 |
| $f/100$ .. | 494 | 475 | 412 | 338 | 270 | 214 | 171 |

16. $v$ cu. ft. is the volume of a quantity of steam when its pressure is $p$ lb./in.$^2$. Law: $pv^n=C$. [Plot $\log p$ against $\log v$.]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $p$ .. | 10 | 25 | 50 | 100 | 200 | 300 | 400 |
| $v$ .. | 5·36 | 2·75 | 1·66 | 1·01 | 0·60 | 0·49 | 0·36 |

17. $t$ sec. is the time taken for a circular disc mounted on an axle to roll down an inclined plane 6 ft. long when the upper end of the plane is $h$ ft. above the lower end. Law: $t=ah^n$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $h$ .. | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 | $1\frac{1}{2}$ | 2 |
| $t$ .. | 22·6 | 15·9 | 11·3 | 9·2 | 8·0 | 6·5 | 5·6 |

18. I candles is the luminosity of a metal filament lamp at V volts. Law: $I=kV^n$.

| | | | | | | |
|---|---|---|---|---|---|---|
| I .. .. | 21·3 | 35·5 | 56·3 | 89·1 | 128·8 | 186 |
| V .. .. | 70 | 80 | 90 | 100 | 110 | 120 |

# GEOMETRY

## CHAPTER VI

## SIMILAR FIGURES

**Some facts about parallel lines. Proportional division.**

**Exercise.**—Mark three points $A$, $B$, $C$ on a straight line at equal distances apart, and through these points draw any three parallel straight lines, as in Fig. 38. Now draw a number of other straight lines cutting the three parallel lines in points such as $D$, $E$, $F$; $L$, $M$, $N$, etc.

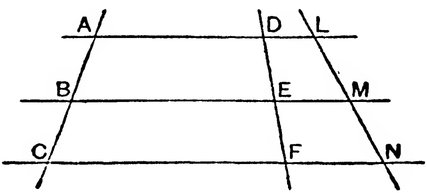



FIG. 38.

Measure $DE$ and $EF$; $LM$ and $MN$; etc. What do you notice?

The student will find that $DE = EF$; $LM = MN$, etc. We may express this result as follows:

**Theorem.**—If three parallel lines cut off equal lengths on any one transversal they also cut off equal lengths on any other transversal.

We shall now *prove* this theorem.

Draw $DH$ and $EK$ parallel to the line $ABC$ (Fig. 39). Then $ADHB$ is a parallelogram and so $DH = AB$. Also, $BEKC$ is a parallelogram, and so $EK = BC$.

But $AB = BC$, whence $DH = EK$.

Thus, in $\triangle$'s $DHE$ and $EKF$,

$$DH = EK,$$

$$\hat{\alpha} = \hat{\beta} \text{ (since } BE \text{ is parallel to } CF\text{)}$$

$$\hat{\gamma} = \hat{\delta} \text{ (since } DH \text{ is parallel to } EK\text{)}$$

$$\therefore\ \triangle DHE \equiv \triangle EKF \text{ (two } \angle\text{'s and a side).}$$

Hence $$DE = EF.$$

It is easily seen that this theorem is also true if we have more than three parallel lines. For example, in Fig. 40, if $AB = BC = CD = DE$ then it follows that $A'B' = B'C' = C'D' = D'E'$.

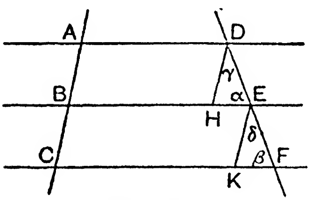



Fig. 39.

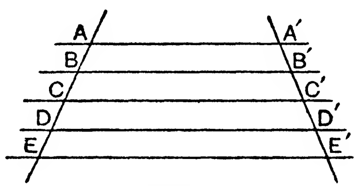



Fig. 40.

**Construction.**—To divide a given straight line into any number of equal parts, without measurement or calculation.

Suppose we wish to divide the line $AB$ into five equal parts. Through $A$ draw any line $AC$ of indefinite length, and *with a pair of compasses* mark off on $AC$ five equal lengths $AD$, $DE$, $EF$, $FG$, $GH$, of any convenient size (Fig. 41).

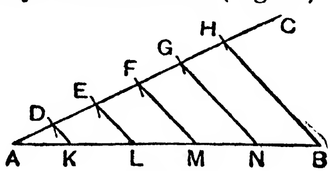



Fig. 41.

Join the last point, $H$, to $B$, and through $D$, $E$, $F$, $G$ draw straight lines parallel to $HB$ (by means of a ruler and set square). The points $K$, $L$, $M$, $N$, in which the lines cut $AB$, are the required points of division.

We have here divided $AB$ into five equal parts, but the method of construction is the same whatever the number of parts.

**Exercise.**—Mark off three points $A$, $B$, $C$ on a straight line such that $AB$ is twice the length of $BC$. Through $A$, $B$, $C$ draw any three parallel straight lines (Fig. 42). Now draw a number of other straight lines cutting these parallels in points such as $D$, $E$, $F$; $L$, $M$, $N$, etc.

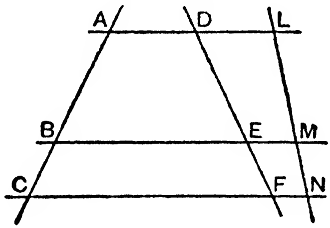



FIG. 42.

Measure $DE$ and $EF$, $LM$ and $MN$, etc., and find the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc. What do you notice?

**Exercise.**—Repeat the previous construction making $AB$ three times the length of $BC$. What do you notice about the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc., in this case?

**Exercise.**—Repeat the construction, making the ratio $\frac{AB}{BC}=\frac{2}{5}$, and state your conclusion.

The student will find that, in every case, the ratios $\frac{DE}{EF}$, $\frac{LM}{MN}$, etc. are each equal to the ratio $\frac{AB}{BC}$.

We therefore conclude the following:

**Theorem.**—The ratios of the intercepts made by three given parallel lines on all transversals are the same.

To *prove* this, suppose that $\frac{AB}{BC}=\frac{m}{n}$, where $m$ and $n$ are whole numbers. Divide $AB$ into $m$ equal parts, and $BC$ into $n$ equal parts; then all the $m+n$ parts into which $AC$ is divided are equal. Through the points of division draw lines parallel to $AD$, as shown by the dotted lines in Fig. 43. These lines cut $DF$ into $m+n$ equal parts, of which $DE$ contains $m$ and $EF$ contains $n$. (In Fig. 43 $m=7$ and $n=5$.)

$$\therefore \frac{DE}{EF}=\frac{m}{n} \qquad \therefore \frac{AB}{BC}=\frac{DE}{EF}.$$

In Fig. 43 $DEF$ is any transversal. If we draw it to pass through $A$, we obtain Fig. 44, in which $D$ coincides with $A$. The preceding theorem now tells us that, if $BE$ is parallel to the side $CF$ of a triangle $ACF$, then $\frac{AB}{BC}=\frac{AE}{EF}$.

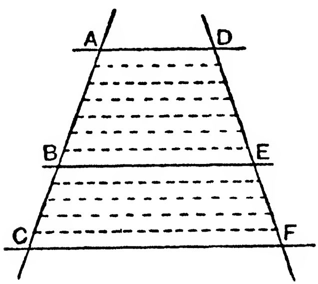



FIG. 43.

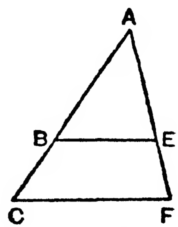



FIG. 44.

Thus we have the following:

**Theorem.**—If a line is drawn parallel to one side of a triangle, it divides the other two sides proportionally.

The converse of this theorem is also true, viz.:

**Theorem.**—If a line divides two sides of a triangle proportionally, it is parallel to the third side.

To prove this, suppose we are given that $DE$ divides the sides $AB$, $AC$ of the $\triangle ABC$ proportionally; i.e. that $\frac{AD}{DB}=\frac{AE}{EC}$.

We have to prove that $DE$ is parallel to $BC$.

If it is not, draw the line $DF$, through $D$, parallel to $BC$.

Then, from the previous theorem, $\frac{AD}{DB}=\frac{AF}{FC}$.

Hence $\frac{AE}{EC}=\frac{AF}{FC}$, and therefore $E$ coincides with $F$. But $DF$ was drawn parallel to $BC$; therefore $DE$ is parallel to $BC$.

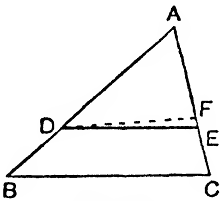


Fig. 45.

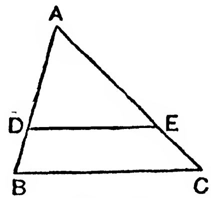


Fig. 46.

As *particular cases* of these two theorems we have the following:

(1) The line through the mid-point of one side of a triangle parallel to the base bisects the other side.

(2) The line joining the mid-points of two sides of a triangle is parallel to the third side.

Note.—There is another way of stating the result expressed in the theorem at the foot of p. 123. If $DE$ is parallel to the side $BC$ of a triangle $ABC$ (Fig. 46) we proved there that

$$\frac{AD}{DB}=\frac{AE}{EC}$$

$$\therefore \frac{DB}{AD}=\frac{EC}{AE}$$

$$\therefore 1+\frac{DB}{AD}=1+\frac{EC}{AE}$$

$$\therefore \frac{AD+DB}{AD}=\frac{AE+EC}{AE}$$

i.e. $$\frac{AB}{AD}=\frac{AC}{AE} \quad \text{or} \quad \frac{AD}{AB}=\frac{AE}{AC}.$$

This form of the result is often useful.

**Internal and external division**

Suppose that $A$, $B$ are two points on a line, 5 in. apart.

Mark the point $P$, between $A$ and $B$, 3 in. from $A$, and the point $Q$, in $AB$ produced, 15 in. from $A$ (Fig. 47).

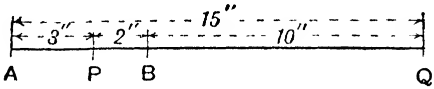



FIG. 47.

Then $PB=2$ in., and $QB=10$ in. and hence $\frac{AP}{PB}=\frac{3}{2}$ and $\frac{AQ}{QB}=\frac{15}{10}=\frac{3}{2}$.

The two points $P$ and $Q$ both divide $AB$ in the ratio $3:2$. We say that "$P$ divides $AB$ internally in the ratio $3:2$," and that "$Q$ divides $AB$ externally in the ratio $3:2$."

If we take *any* ratio, we shall find that there are always two points which divide $AB$ in that ratio, one of the points dividing $AB$ internally, the other externally.

*Example.*—$A$, $B$ are two points 12 cm. apart. Find, by calculation, the positions of the two points which divide $AB$ in the ratio $3:5$.

Let $P$ be the point of internal division, and suppose $AP=x$ cm. Then $PB=(12-x)$ cm.

$$\therefore \frac{x}{12-x}=\frac{3}{5}$$

$$5x = 3(12 - x)$$
$$= 36 - 3x$$
$$8x = 36$$
$$\therefore x = \tfrac{36}{8} = \tfrac{9}{2} = 4{\cdot}5$$

$\therefore$ $P$ is 4·5 cm. from $A$, between $A$ and $B$.

[Check: $AP = 4{\cdot}5$ cm. $\therefore PB = 7{\cdot}5$ cm. $\therefore \dfrac{AP}{PB} = \dfrac{4{\cdot}5}{7{\cdot}5} = \tfrac{9}{15} = \tfrac{3}{5}$.]

Let $Q$ be the point of external division, and let $AQ = y$ cm. Since $\dfrac{AQ}{QB} = \tfrac{3}{5}$, $AQ$ must be less than $QB$, and hence $Q$ must be nearer to $A$ than to $B$. Thus $Q$ must lie outside $AB$ beyond $A$ (see Fig. 48).

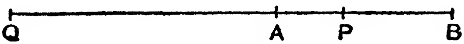



FIG. 48.

$$\therefore QB = (y + 12) \text{ cm.}$$
$$\therefore \frac{y}{y+12} = \tfrac{3}{5}$$
$$5y = 3(y + 12)$$
$$= 3y + 36$$
$$2y = 36$$
$$\therefore y = 18.$$

$\therefore$ $Q$ is 18 cm. from $A$, outside $AB$.

[Check: $AQ = 18$ cm., $\therefore QB = 30$ cm., $\therefore \dfrac{AQ}{QB} = \tfrac{18}{30} = \tfrac{3}{5}$.]

**Exercise.**—Verify that the theorems on pp. 123 (at foot) and 124 are true whether the line divides the sides internally or externally.

A *graphical* construction for dividing a line in a given ratio is given below.

**Construction.**—To divide a given straight line in a given ratio, internally or externally, without measurement or calculation.

Suppose, for example, we wish to divide the line $AB$ in the ratio 7 : 2.

(i) *Internally.*—Draw any line $AC$ through $A$, and with a pair of compasses mark off seven equal lengths from $A$, terminating at $D$ (Fig. 49). From $D$ mark off another two of these equal lengths, terminating at $E$. Join $E$ to $B$, and draw $DP$ parallel to $EB$. Then $\frac{AP}{PB}=\frac{AD}{DE}=\frac{7}{2}$.

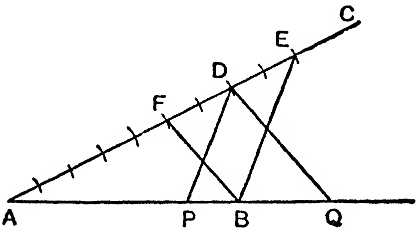



Fig. 49.

(ii) *Externally.*—In this case mark off the two equal lengths from $D$ *backwards* towards $A$, terminating at $F$. Join $F$ to $B$, and draw $DQ$ parallel to $FB$. Then $\frac{AQ}{QB}=\frac{AD}{DF}=\frac{7}{2}$.

The above constructions apply whatever the ratio $m:n$ in which it is required to divide $AB$. If $m$ is less than $n$, however, the point $F$ will be on the opposite side of $A$ to $D$, as in Fig. 50.

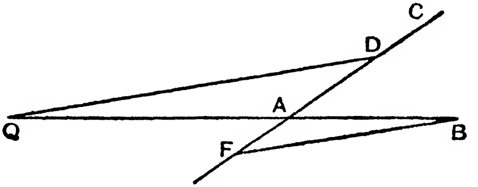



Fig. 50.

**Definition.**—If $a$, $b$, $c$, $d$ are four magnitudes such that $\frac{a}{b}=\frac{c}{d}$, then $d$ is called the *fourth proportional* to $a$, $b$, $c$.

**Construction.**—To find, graphically, the fourth proportional to three given lengths.

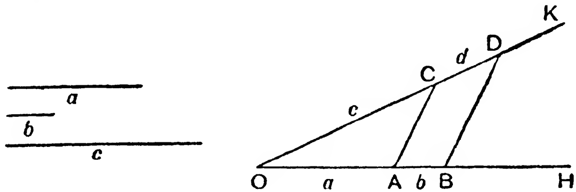



FIG. 51.

Let $a$, $b$, $c$ be the three given lengths.

Draw any two intersecting lines $OH$, $OK$.

Along $OH$ mark off $OA=a$, $AB=b$, and along $OK$ mark off $OC=c$.

Join $AC$ and, through $B$, draw $BD$ parallel to $AC$, cutting $OK$ in $D$.

Then $CD$ is the fourth proportional to $a$, $b$, $c$.

The proof is left to the student.

**Theorem.**—The internal and external bisectors of an angle of a triangle divide the opposite side, internally and externally, in the ratio of the sides containing the angle.

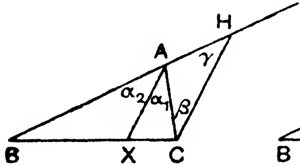



FIG. 52 (*a*).

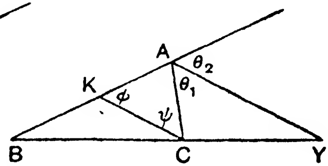



FIG. 52 (*b*).

Let $ABC$ be the triangle and $AX$, $AY$ the internal and external bisectors of the angle $A$.

In Fig. 52 ($a$), draw $CH$ parallel to $XA$ cutting $BA$ produced in $H$.

Then

$\widehat{\beta}=\widehat{\alpha_1}$ (alternate angles)

$\widehat{\gamma}=\widehat{\alpha_2}$ (corresponding angles)

But $\widehat{\alpha_1}=\widehat{\alpha_2}$

$\therefore\ \widehat{\beta}=\widehat{\gamma}$

$\therefore\ \triangle ACH$ is isosceles

$\therefore\ AC=AH.$

Since $AX$ is parallel to $HC$,

$$\frac{BX}{XC}=\frac{BA}{AH}$$

$$\therefore\ \frac{BX}{XC}=\frac{BA}{AC}$$

In Fig. 52 ($b$), draw $CK$ parallel to $YA$ cutting $BA$ in $K$.

Then

$\widehat{\psi}=\widehat{\theta_1}$ (alternate angles)

$\widehat{\phi}=\widehat{\theta_2}$ (corresponding angles)

But $\widehat{\theta_1}=\widehat{\theta_2}$

$\therefore\ \widehat{\psi}=\widehat{\phi}$

$\therefore\ \triangle AKC$ is isosceles.

$\therefore\ AC=AK.$

Since $AY$ is parallel to $KC$,

$$\frac{BY}{YC}=\frac{BA}{AK}$$ (see exercise, p. 126.)

$$\therefore\ \frac{BY}{YC}=\frac{BA}{AC}$$

## Exercise XIX

1. Draw a line about 4 in. long, and divide it into seven equal parts by a graphical construction.

2. Draw a line about 5 in. long, and divide it *graphically* in the ratio 4 : 3 (i) internally, (ii) externally.

3. Draw a line $AB$ of length 2·6 in. Find, by a graphical construction, the points in $AB$ (produced if necessary) such that $AP=3PB$. Check by calculation.

4. Find, *graphically*, the fourth proportional to 3, 7, 2. Check by calculation.

5. [If $a:b=b:x$, then $x$ is called the *third proportional* to $a$ and $b$.] Find, graphically, the third proportional to 4·5 and 3·1.

6. If $D$, $E$, $F$ are the mid-points of the sides of the triangle $ABC$ (Fig. 53), prove that the four triangles shown are all congruent to one another.

7. Deduce, from Question 6, the following theorem: The line joining the mid-points of two sides of a triangle is equal in length to half the third side.

8. Find a construction for drawing a triangle when only the mid-points of its sides are given.

9. If $ABCD$ is any quadrilateral (Fig. 54) and $P$, $Q$, $R$, $S$, are the mid-points of the sides taken in order, prove that $PQRS$ is a parallelogram. [Hint.—Draw the diagonals $AC$, $BD$.]

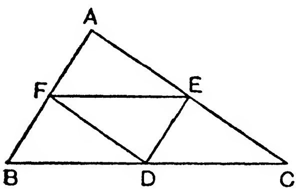



FIG. 53.

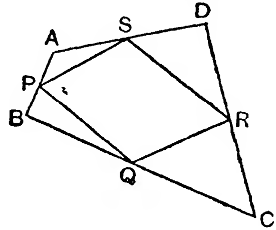



FIG. 54.

10. A ladder, 18 ft. long, rests over a wall which is 6 ft. high, the foot of the ladder being on the ground and its upper end resting against the side of a house. If the foot of the ladder is 8 ft. from the bottom of the wall, find the distance between the wall and the house. Verify by a scale drawing.

11. In Fig. 55 $ABCD$ is a trapezium, and $M$, $N$ are the mid-points of $AD$, $BC$. Prove that $MN$ is parallel to $AB$ and $DC$. [Compare the proof of the theorem on p. 124.]

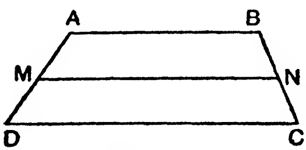



FIG. 55.

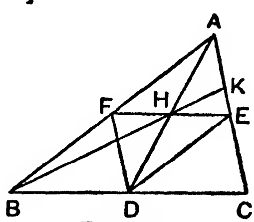



FIG. 56.

12. In Fig. 56 $D$, $E$, $F$ are the mid-points of the sides of the triangle $ABC$. The lines $AD$, $EF$ intersect in $H$ and the line $BH$ cuts $AC$ in $K$.

Prove that (i) $H$ is the mid-point of $AD$; (ii) $K$ is a point of trisection of $AC$.

[Hint for (ii): Draw the line through $D$ parallel to $BK$.]

13. $P$ is any point in the side $AB$ of a triangle $ABC$ (Fig. 57). $PQ$ is parallel to $BC$, and $QR$ is parallel to $AB$. Prove that $\frac{BR}{RC}=\frac{AP}{PB}$.

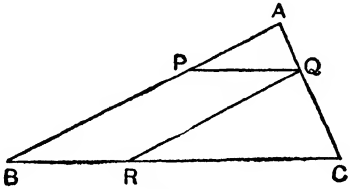



FIG. 57.

14. From the theorem on p. 128 devise a construction for dividing a line internally and externally in a given ratio.

15. Prove that the locus of a point which moves so that the ratio of its distances from two fixed points is constant, is a circle.

[Hint.—Use the theorem on p. 128.]

## Similarity

Figures which are of the same shape but different in size are said to be *similar*.

When a photograph is enlarged any object in the picture retains its shape but is magnified in size. The object and its enlargement are similar figures.

A plan of a field is similar to the actual field. Two maps of a town, drawn on different scales, are similar.

We can also have similar solids. For example, in a model of a ship all the parts are reproduced to their proper shape, but on a smaller scale. The model is *similar* to the actual ship.

The principle of similarity is very important. Models are widely used for testing the behaviour of new designs of aeroplanes and ships in the experimental stages.

In similar figures it is clear that angles are unaltered, and although lengths are altered they are all increased or decreased in the same ratio, so that the *ratio* of the lengths of any two lines is unaltered.

A *polygon* is a figure bounded by any number of straight lines.

If two polygons are similar, they must have the same number of sides and the angles of one polygon must be equal to the angles of the other, *taken in the same order* round the figure. For example, in Fig. 58, $ABCDE$ and $A'B'C'D'E'$ are similar, and $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, etc.

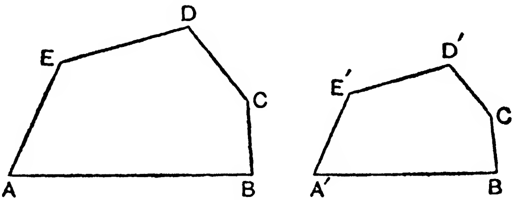



Fig. 58.

Also, since all lengths are altered in the same ratio in passing from one figure to the other, it follows that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'};$$

that is, the sides of one polygon must be proportional to the corresponding sides of the other polygon.

It is important to notice that two polygons may be equiangular (i.e. have their angles equal taken in the same order) and yet not be similar, since the angles alone do not determine the shape of a figure. This is seen in Fig. 59, where $HK$ is parallel to $CD$; the figures $ABCD$ and $ABHK$ have the same angles but are obviously not of the same shape. Neither is it sufficient for similarity that the sides of the two figures should be proportional, since the lengths of the sides alone do not determine the shape of a polygon. We can see this

from Fig. 60, where $ABCD$ and $A_1BCD_1$ are two quadrilaterals whose sides are of the same lengths. (If we regard $ABCD$ as a frame made up of four rods pin-jointed at their ends, we can deform the frame into the shape $A_1BCD_1$ without altering the lengths of the rods.)

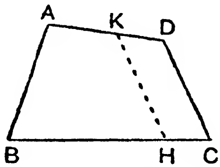


FIG. 59.

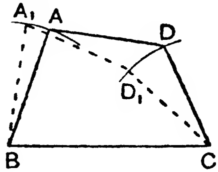


FIG. 60.

Thus, for two polygons of the same number of sides to be similar, *two* conditions must be satisfied, viz.:

(i) they must be equiangular.

(ii) their corresponding sides must be proportional in length.

## Similar triangles

A triangle, which is the simplest of all polygons, is different from all other polygons in one important respect; namely, that its angles determine its shape, so that if two triangles are equiangular they are similar.

Thus if two triangles are equiangular their corresponding sides *are* proportional; the converse is also true, that if the sides of two triangles are proportional the triangles are equiangular.

These two statements are so important that we shall prove them, but the student is first advised to work the following exercises.

**Exercise.**—Draw two triangles of different sizes having their angles equal to 25°, 70°, 85°, and measure their sides. Compare the ratios of the pairs of corresponding sides of the two triangles.

**Exercise.**—Draw two triangles having sides of lengths 4 cm., 6 cm., 2·5 cm., and 8 cm., 12 cm., 5 cm. Measure the angles of the two triangles, and verify that their corresponding angles are equal.

[NOTE.—"Corresponding angles" here means "angles opposite corresponding sides."]

We shall now prove the two theorems stated above.

**Theorem.**—If two triangles are equiangular, their corresponding sides are proportional.

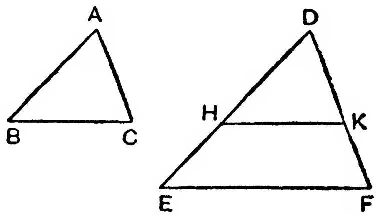



FIG. 61.

Let $ABC$, $DEF$ be the triangles, with $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$.

We have to prove that $\dfrac{BC}{EF} = \dfrac{CA}{FD} = \dfrac{AB}{DE}$.

On $DE$ mark off $DH$ equal to $AB$, and on $DF$ mark off $DK$ equal to $AC$. Join $HK$.

Then in $\triangle$'s $ABC$ and $DHK$,

$$AB = DH,\ AC = DK,\ \angle A = \angle D,$$

$\therefore\ \triangle ABC \equiv \triangle DHK$ (two sides and included angle)

$\therefore\ \angle DHK = \angle B = \angle E.$

$\therefore\ HK$ is parallel to $EF$

$$\therefore\ \frac{DH}{DE} = \frac{DK}{DF} \text{ (p. 124)}$$

$$\therefore\ \frac{AB}{DE} = \frac{AC}{DF}.$$

In the same way, by marking off lengths on $ED$ and $EF$ equal to $BA$ and $BC$, we can prove that $\frac{AB}{DE}=\frac{BC}{EF}$.

Hence $$\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}.$$

**Theorem.**—If the sides of two triangles are proportional, their corresponding angles are equal.

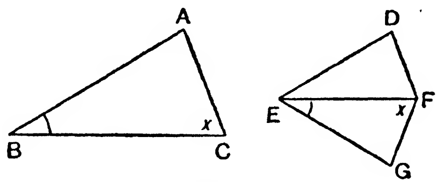



Fig. 62.

Let $ABC$, $DEF$ be the triangles, in which

$$\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}.$$

We have to prove that $\angle A=\angle D$, $\angle B=\angle E$ and $\angle C=\angle F$. (Note.—Angles $A$ and $D$ are corresponding angles, since they are opposite the corresponding sides $BC$ and $EF$.)

Draw a triangle $EFG$, on the opposite side of $EF$ to the triangle $DEF$, having $\angle FEG=\angle B$ and $EFG=\angle C$. Then $\triangle$'s $ABC$ and $GEF$ are equiangular, and hence (from the previous theorem) $\frac{BC}{EF}=\frac{CA}{FG}=\frac{AB}{GE}$.

But $\frac{BC}{EF}=\frac{CA}{FD}=\frac{AB}{DE}$, and hence, by comparing these ratios, $FG=FD$ and $GE=DE$.

Thus the $\triangle$'s $DEF$ and $GEF$ have their three sides equal, and therefore they are congruent.

$\therefore \angle D=\angle G$, $\angle DEF=\angle FEG$ and $\angle DFE=\angle EFG$.

It follows that $\angle D=\angle A$, $\angle DEF=\angle B$ and $\angle DFE=\angle C$.

[*Note.*—In this proof the triangle $EFG$ was drawn on the *opposite* side of $EF$ to $D$ merely in order to obtain a clear figure, since if it were drawn on the same side as $D$ it would coincide with the triangle $EFD$.]

The following theorem will sometimes be found useful. Its proof is left as an exercise for the student, but the main steps of the proof are given.

**Theorem.**—If two triangles have one angle of one equal to one angle of the other and the sides containing those angles proportional, the triangles are similar.

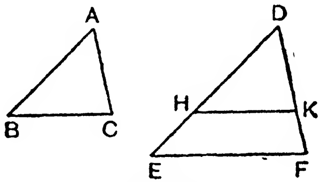



FIG. 63.

Suppose we are given $\triangle$'s $ABC$, $DEF$ in which $\angle A = \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

Mark off $DH = AB$, $DK = AC$. Show that $HK$ is parallel to $EF$. Then show that each of the triangles $ABC$, $DEF$ is equiangular to $\triangle DHK$. It follows that the triangles $ABC$, $DEF$ are themselves equiangular and therefore similar.

## Diagonal scale

By means of a diagonal scale we can measure distances more accurately than would be possible on an ordinary scale. For example, it is possible to calibrate an ordinary scale, and to read it, to tenths of an inch with ease, but to calibrate to hundredths of an inch would require much greater care and expense, and, even if it were so calibrated, to read the divisions would not be easy without a magnifying lens. By means of a

diagonal scale we can measure distances to a hundredth of an inch. The scale is usually drawn on hardwood, ivory or celluloid. It is shown in Fig. 64.

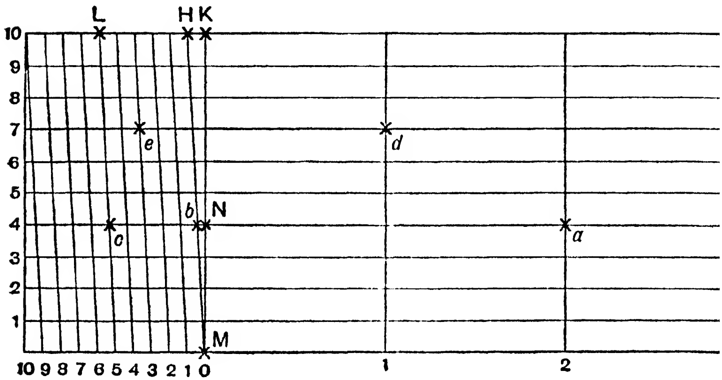



FIG. 64.

Each of the large divisions along the top and bottom of the scale = 1 in.

Each of the small divisions along the top and bottom of the scale $= \frac{1}{10}$ in.

The horizontal lines are at equal distances apart.

The triangles $MbN$, $MHK$ are equiangular and therefore similar.

$$\therefore \frac{bN}{HK} = \frac{MN}{MK} = \frac{4}{10}.$$

$$\therefore bN = \frac{4}{10} HK = \frac{4}{10} \times \frac{1}{10} \text{ in.} = \frac{4}{100} \text{ in.}$$

$$\therefore ab = aN + Nb = (2 + \tfrac{4}{100}) \text{ in.} = 2 \cdot 04 \text{ in.}$$

It is clear that the oblique lines, such as $bH$ and $cL$, are all parallel, and the lengths cut off between two consecutive oblique lines on all the horizontal lines are equal, being equal to $\frac{1}{10}$ in. Thus $bc$ measures $5 \times \frac{1}{10}$ in., i.e. $\frac{5}{10}$ in.

Hence $ac = ab + bc = (2 + \frac{4}{100} + \frac{5}{10})$ in. $= 2 \cdot 54$ in.

5*

To read off a length of 2·54 in. the procedure is as follows :

Place the point of a pencil on the figure 2 on the bottom scale to the *right* of $O$, and the point of another pencil on the figure 5 on the bottom scale to the *left* of $O$. Now move both pencils up the vertical, or oblique, lines through those points as far as the horizontal line through the point marked 4 on the vertical scale at the left. The distance between the two pencil points will then be 2·54. This length can then be transferred to paper if required by means of a pair of dividers. Similarly for any other length.

**Exercise.**—1. What is the length of *de* in Fig. 64?

2. Mark a distance 3·72 in. on the scale in Fig. 64.

## Similar polygons

**Construction.**—To construct a polygon similar to a given polygon and on a given scale.

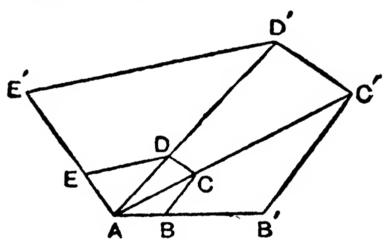



FIG. 65.

Suppose, for example, we wish to draw a polygon similar to $ABCDE$ (Fig. 65) but having its sides three times the lengths of those of the given polygon.

Produce $AB$ and mark off $AB' = 3 \, . \, AB$.

Through $B'$ draw $B'C'$ parallel to $BC$, cutting $AC$ produced in $C'$.

Through $C'$ draw $C'D'$ parallel to $CD$, cutting $AD$ produced in $D'$.

Through $D'$ draw $D'E'$ parallel to $DE$, cutting $AE$ produced in $E'$.

Then $AB'C'D'E'$ is the required polygon.

Proof : The two polygons are obviously equiangular since the sides of the second polygon have been drawn parallel to those of the first.

We have to prove, in addition, that their corresponding sides are proportional.

Since $B'C'$ is parallel to $BC$, $\triangle$'s $ABC$ and $AB'C'$ are similar.

$$\therefore \frac{B'C'}{BC}=\frac{AB'}{AB}=\frac{3}{1}. \quad \text{Also } \frac{AC'}{AC}=\frac{AB'}{AB}=\frac{3}{1}.$$

Since $C'D'$ is parallel to $CD$, $\triangle$'s $ACD$ and $AC'D'$ are similar.

$$\therefore \frac{C'D'}{CD}=\frac{AC'}{AC}=\frac{3}{1}. \quad \text{Also } \frac{AD'}{AD}=\frac{AC'}{AC}=\frac{3}{1}.$$

Finally, since $D'E'$ is parallel to $DE$, $\triangle$'s $ADE$ and $AD'E'$ are similar.

$$\therefore \frac{D'E'}{DE}=\frac{AE'}{AE}=\frac{AD'}{AD}=\frac{3}{1}.$$

Thus each side of the polygon $AB'C'D'E'$ is three times the length of the corresponding side of the polygon $ABCDE$.

The method of construction is the same whatever the required ratio of the sides. Thus, if the sides of the new polygon are to be $n$ times the length of those of the given polygon, we make $AB'=n\,.\,AB$ and then proceed as above. ($n$ may be greater than, or less than, 1.)

An *alternative construction* is as follows :

Take any point $O$ and join it to each of the vertices of the given polygon $ABCDE$.

(The point $O$ may be inside or outside the given polygon ; see Figs. 66 and 67.)

On $OA$, produced if necessary, mark off $OA'=n\,.\,OA$.

Through $A'$ draw $A'B'$ parallel to $AB$ cutting $OB$ (produced if necessary) in $B'$. Through $B'$ draw $B'C'$ parallel to $BC$ cutting $OC$ (produced if necessary) in $C'$ ; and so on.

Then $A'B'C'D'E'$ is the required polygon.

The proof is left as an exercise to the student. It is very like that for the previous construction; in fact, the student will notice that the previous construction is only the special case of that just given when $O$ is taken at $A$.

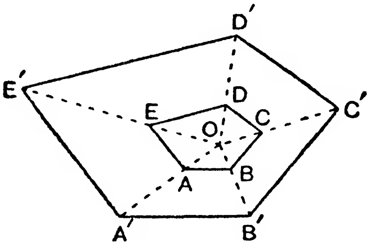



FIG. 66.

Figures, such as $ABCDE$ and $A'B'C'D'E'$ in Figs. 66 and 67, which are so placed that the lines joining corresponding points are concurrent are said to be "*in perspective.*" The point of concurrence, which, in Figs. 66 and 67, is the point $O$, is called the *centre of perspective.*

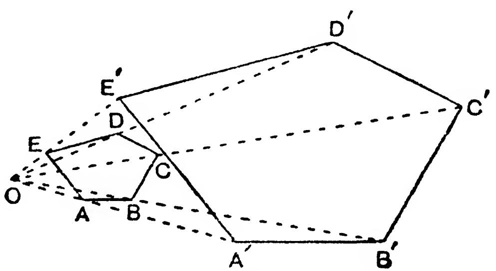



FIG. 67.

Figures in two different planes may be in perspective. For example, in the case of a pin-hole camera, the object and its image are in perspective, the centre of perspective being the pin-hole (Fig. 68).

NOTE.—If two figures are in perspective they are not necessarily similar. They will be so when they are in parallel planes, *or*, if in the same plane, when their corresponding sides are parallel.

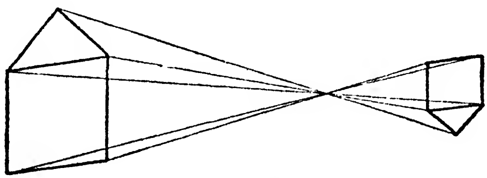


FIG. 68.

**Similar irregular figures**

We can construct a figure similar to a given figure, whatever its shape, on any desired scale by using the method of perspective.

For example, to draw a figure similar to that shown on the left in Fig. 69, but of one-quarter the scale, take any point $O$, join it to a point $A$ of the figure and mark off $OA' = \frac{1}{4}OA$. Then join $O$ to another point $B$ of the figure and mark off $OB' = \frac{1}{4}OB$, and so on for a large number of points. Join up the points $A'$, $B'$, etc., by a freehand curve. The larger the number of points taken, of course, the better will be the resulting figure.

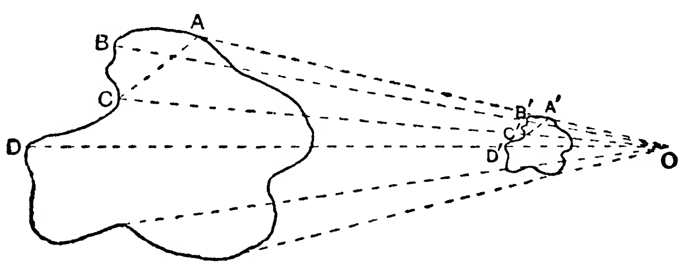



FIG. 69.

The length of any line in the second figure will be one-quarter that of the corresponding line in the first figure; for example, $\frac{A'C'}{AC} = \frac{OA'}{OA} = \frac{1}{4}$.

[Such constructions can be performed mechanically by means of an instrument called a *pantograph*, the theory of which is indicated in Exercise XX, Question 23 (p. 148).]

**Areas of similar figures**

Two rectangles whose sides are 2 in., 3 in., and 4 in., 6 in., respectively, are similar, since they are equiangular and the sides of the second rectangle are twice the lengths of the sides of the first rectangle. Their areas are 6 sq. in. and 24 sq. in., respectively, so that the area of the second is four times that of the first.

Generally, if the sides of the first rectangle are of lengths $a$ and $b$, and those of the second $na$ and $nb$, the ratio of the areas of the rectangles is $\frac{na \times nb}{a \times b} = \frac{n^2ab}{ab} = \frac{n^2}{1}$.

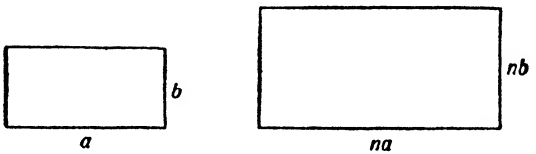



FIG. 70.

Thus the ratio of the areas of similar rectangles is equal to the *square* of the ratio of corresponding sides.

Any area, with an irregular or curved boundary, can be regarded as built up of rectangles. For if we inscribe rectangles in the area and then draw smaller and smaller rectangles in the spaces which remain near the boundary, as shown in Figs. 71 ($a$) or 71 ($b$), the rectangles will cover more and more of the area each time, and will tend ultimately to cover the whole area.

If the figure is enlarged, or reduced, a similar figure is obtained. Every line in the figure is enlarged, or reduced, in the same ratio. Fig. 71 ($b$) is an enlargement of Fig. 71 ($a$). If every line in Fig. 71 ($b$) is $n$ times the length of the corresponding line in Fig. 71 ($a$) (e.g. $A'B' = n \, . \, AB$), the area of each of the rectangles in Fig. 71 ($b$) is $n^2$ times the area of the corresponding rectangle in Fig. 71 ($a$). Thus by adding up the rectangles we see that the total area in Fig. 71 ($b$) is $n^2$ times the total area in Fig. 71 ($a$).

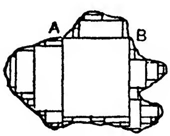



FIG. 71 ($a$).

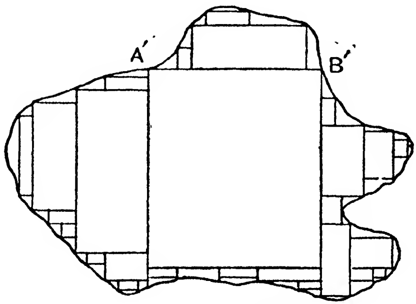



FIG. 71 ($b$).

Hence the ratio of the areas of two similar figures is equal to the square of the ratio of the lengths of corresponding lines in the two figures.

We usually express this briefly by saying that *the areas of similar figures are proportional to the squares of their corresponding linear dimensions.*

**Example.**—Two scale drawings of a machine are made, one on a scale of 2 ft. to 1 in., the other on a scale of 3 ft. to 1 in. If the area of a certain plate on the first drawing is 1·64 sq. in. what is its area on the second drawing ?

An actual length of 1 ft. is represented by $\frac{1}{2}$ in. on the first drawing and by $\frac{1}{3}$ in. on the second drawing.

$$\therefore \frac{\text{Area on second drawing}}{\text{Area on first drawing}} = \left(\frac{\frac{1}{3}\text{ in.}}{\frac{1}{2}\text{ in.}}\right)^2 = \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9}.$$

$\therefore$ Area on second drawing $= \frac{4}{9} \times$ area on first drawing.

$$= \frac{4}{9} \times 1{\cdot}64 \text{ sq. in.}$$

$$= 0{\cdot}73 \text{ sq. in.}$$

## Exercise XX

**1.** State whether the following pairs of figures are similar or not (if not, give your reasons):

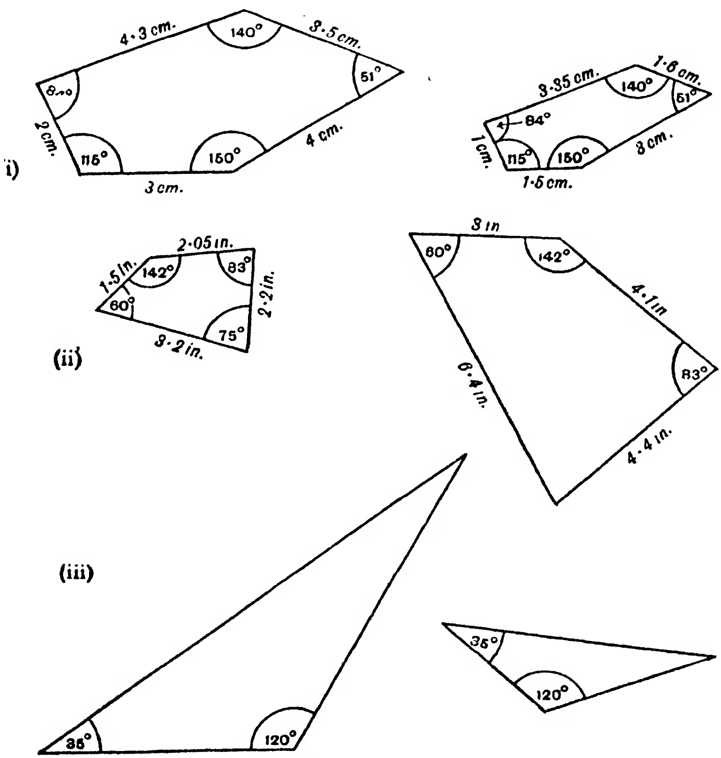
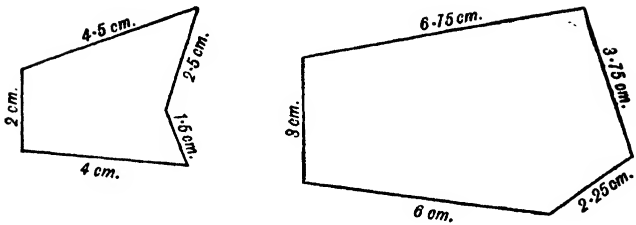
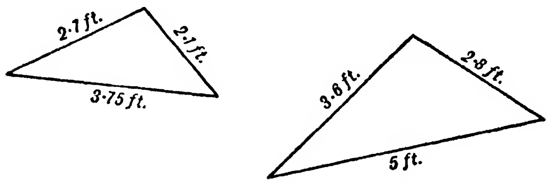



Fig. 72.

(iv)




(v)




(vi)

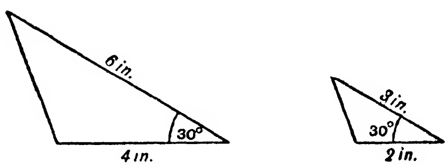



Fig. 72—*continued.*

2. The sides of a quadrilateral are 6 in., 4 in., 5 in., 7 in. Find the lengths of the sides of a similar quadrilateral whose shortest side measures 5 cm.

3. Fig. 73 represents a ground plan of a house, the lengths indicated being the lengths as measured on the plan. If the actual length of the side *AB* is 36 ft., find all the other lengths and the actual area of the site.

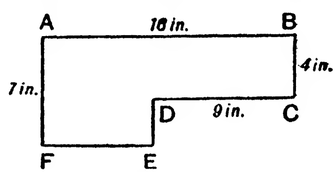



Fig. 73.

4. A locomotive engine, which runs on a track 4 ft. $8\frac{1}{2}$ in. wide, is 24 ft. long and the diameter of its driving wheels is 5 ft. 6 in. If

a model is to be constructed to run on a track 3 in. wide, what must be its length and the diameter of its wheels ?

**5.** A road has a constant gradient. After travelling half a mile a man finds that he has risen 150 ft. How much farther must he travel before he has risen a total of 400 ft. ?

**6.** The shadow of a vertical pole 8 ft. high cast by the sun is 10 ft. long, and at the same time the shadow of a chimney stack is 95 ft. long. What is the height of the stack?

**7.** Find the length of the shadow of a man 5 ft. 10 in. high standing 9 ft. from the foot of a street lamp which is 17 ft. above the ground.

**8.** A halfpenny (diameter 1 in.) held at 9 ft. $7\frac{3}{4}$ in. from the eye just covers the moon. If the moon's diameter is 2160 miles, find its distance from the Earth.

**9.** The legs of a step-ladder are each 8 ft. long and they are connected by a rope 4 ft. long attached to each of the legs at a distance 2 ft. from the foot. Find the distance between the feet when the ladder is fully open.

**10.** In order to measure the height of a wireless mast a man erects a vertical pole 10 ft. high at a distance of 40 yds. from the mast. He then erects another pole 6 ft. high in such a position that the tops of the two poles and of the mast are in line. If the distance between the two poles is 8 ft., find the height of the mast.

Questions 11–14 are to be done on squared paper, using any convenient axes and scale :

**11.** Mark the points $A$, $B$, $C$ whose co-ordinates are (0, 0), (1, 2), (3, 6) respectively. Prove that they lie in a straight line and find the ratio $AB : BC$.

**12.** Calculate the co-ordinates of the point $P$ which divides $BC$ in Question 11 internally in the ratio 3 : 1. Mark the point $P$ in your figure and verify that $BP : PC = 3 : 1$ by measurement.

**13.** Draw the quadrilateral whose vertices, taken in order, are the points $A$ (0, 0), $B$ (3, 2), $C$ (2, 3), $D$ ( – 1, 4). Draw a similar quadrilateral $PQRS$, with $P$, $Q$ at the points (2, 4), (3·5, 5) and with the side $PQ$ corresponding to $AB$. Read off the co-ordinates of $R$ and $S$.

**14.** Draw a hexagon having its vertices at the points (0, 0), (0, 2), (5, 3), (3, 5), (1, 5), ( – 1, 2). Draw a similar hexagon with the vertices corresponding to the first two points at (0, 0), (0, 3·2). Read off the co-ordinates of the remaining four vertices.

**15.** If $ABC$ is a triangle with a right angle at $A$, and $AD$ is perpendicular to $BC$ (Fig. 74), prove that the triangles $ABC$, $DBA$ and $DAC$ are all similar. Deduce that $\frac{BD}{AD}=\frac{AD}{DC}$, and hence that $AD^2=BD \cdot DC$.

[The *mean proportional* (or geometric mean) of two quantities $a$ and $b$ is defined as the quantity $x$, such that $\frac{a}{x}=\frac{x}{b}$, i.e. such that $x^2=ab$, or $x=\sqrt{ab}$. Thus $AD$ is the mean proportional of $BD$ and $DC$.]

**16.** In Fig. 74 show that $AB$ is the mean proportional of $BC$ and $BD$, and that $AC$ is the mean proportional of $BC$ and $CD$.

**17.** From the results of Question 16 deduce a proof of Pythagoras's Theorem.

**18.** Prove that the line joining the mid-points of two sides of a triangle is equal in length to half the third side.

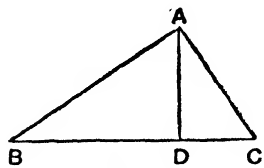



FIG. 74.

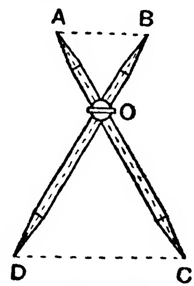



FIG. 75.

**19.** $ABC$ is a triangle; $X$, $Y$, $Z$ are the mid-points of the sides $BC$, $CA$, $AB$ respectively. If $AX$ and $BY$ intersect in $G$, prove that $AG : GX = BG : GY = 2 : 1$. [*Hint.*—Join $XY$ and use the result of Question 18.] Deduce that $CZ$ also passes through $G$.

**20.** $ABCD$ is a trapezium in which $AB$ and $DC$ are the parallel sides. If the diagonals intersect in $O$, prove that $AO : OC = BO : OD$.

**21.** *Proportional compasses.*—In Fig. 75 $AC$, $BD$ are two metal rods with pointed ends and slotted in the middle. A screw $O$ can be moved along the slots and fixed in any desired position. Show that, for a fixed position of the screw, the ratio $AB : DC$ is constant whatever the angle between the legs.

22. If in Fig. 75, $AC = BD = 6$ in., and the screw is set at a distance of $2\frac{1}{2}$ in. from $A$ and $B$, what is the ratio $DC : AB$ ?

Find also the position of the screw for which $DC = 3AB$.

23. *Pantograph.*—A pantograph is a device, in the form of a linkage, for copying any figure on an enlarged or reduced scale. One type of instrument is shown in Fig. 76. $OABC$ is a rhombus of four rods freely hinged at the corners. $PQ$, $RQ$ are two rods freely hinged together at $Q$ and to the original frame at $P$ and $R$, such that $OPQR$ is also a rhombus. A fine steel point is fixed at $Q$ (called the "tracing point") and a pencil at $B$ (called the "scribing point").

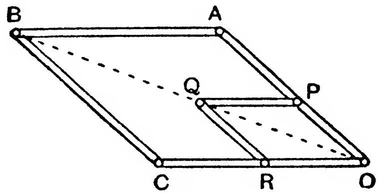



Fig. 76.

Since the diagonals of a rhombus bisect the angles, $A\hat{O}B = \frac{1}{2}A\hat{O}C = P\hat{O}Q$, and thus $O$, $Q$, $B$ are in a straight line.

Prove that triangles $OAB$, $OPQ$ are similar and hence that $\dfrac{OB}{OQ} = \dfrac{OA}{OP}$.

It follows that if $O$ is kept fixed and $Q$ traces out any given figure, the point $B$ will trace out a similar figure.

[The tracing point and scribing point may obviously be interchanged.]

24. Fig. 77 shows another type of pantograph. $AE$, $AB$, $CD$, $CF$ are four rods freely hinged at $A$, $B$, $C$, $D$, so that $AD = CB$

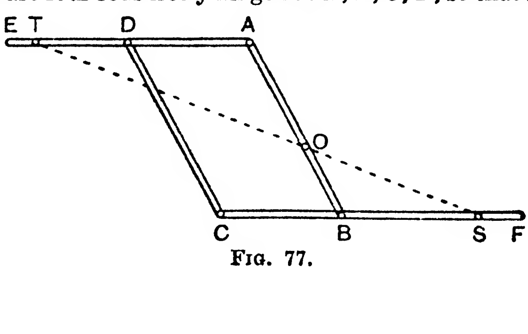



Fig. 77.

and $AB = DC$. The rods $AE$, $CF$, $AB$ are slotted and the tracing point $T$, the scribing point $S$ and a pivot $O$ can be fixed by means of screws anywhere in those rods. $T$, $O$, $S$ are fixed so that they lie in a straight line.

Prove that if $O$ is fixed and $T$ describes a given figure, $S$ describes a similar figure.

If $AD = BC = 6$ in., $AB = 12$ in., $DT = 2$ in., and the scale of the new figure is required to be three times that of the figure traced out by $T$, find the distances of the pivot and the scribing point from $B$.

**25.** The sides of a rectangle are 5 in. and 8 in. long. What are the lengths of the sides of a similar rectangle of four times its area?

**26.** On a certain map an area of 54 sq. ml. is represented by a rectangle 3 in. by $\frac{1}{2}$ in. What is the scale of the map?

**27.** The county of Sussex covers an area of 2·53 sq. in. on a map drawn to a scale of 24 mls. to the inch. What area will it cover on a map having a scale of 4 mls. to the inch?

**28.** A drawing of a certain machine part has the dimensions shown in Fig. 78. Find the area of the drawing. If the drawing is one-quarter actual size, what is the area of the actual part?

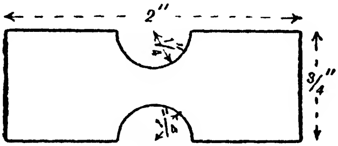



FIG. 78.

**29.** $ABC$ and $A'B'C'$ are similar triangles (Fig. 79); $AD$ is perpendicular to $BC$ and $A'D'$ perpendicular to $B'C'$. Prove

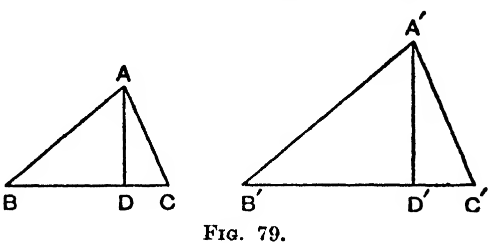



FIG. 79.

that $AD : A'D' = BC : B'C'$. Hence prove that area $\triangle ABC$ : area $\triangle A'B'C' = BC^2 : B'C'^2$; i.e. that the areas of similar triangles are proportional to the squares on corresponding sides.

**30.** A plug fits into the top of a conical hole (Fig. 80). If the thickness of the plug is $t$ and the diameters of its end faces are $a$ and $b$, find the depth of the hole.

**31.** In Fig. 81 the depth of the hole is $d$ and the dimensions of a plug which fits into the top are as shown. Prove that the diameter of the bottom of the hole is $a - \frac{d}{t}(a - b)$.

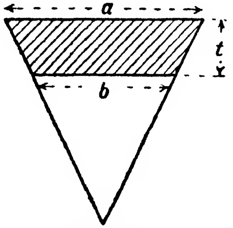



FIG. 80.

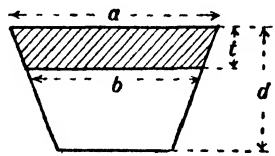



FIG. 81.

**32.** A pair of compasses rests over a cylinder which lies on a table (as in Fig. 82). If the distance $BC$ between the feet of the compasses is 12 cm., and the length ($AB$ or $AC$) of each leg is 10 cm., find the radius of the cylinder.

[Hint.—Prove that triangles $AOE$, $ACD$ are similar.]

**33.** A cone rests over a sphere, as shown in Fig. 83, touching it along a circle of radius $b$. If each point of that circle is a distance $a$ from the vertex of the cone, prove that the radius of the sphere is $\dfrac{ab}{\sqrt{a^2 - b^2}}$.

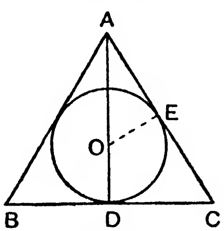



FIG. 82.

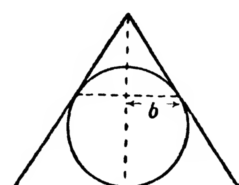



FIG. 83.

**34.** A line $PQ$, parallel to the side $BC$ of a triangle $ABC$, cuts $AB$, $AC$ in $P$, $Q$. A line through $A$ cuts $BC$ in $D$ and $PQ$ in $R$. Prove that $PR : RQ = BD : DC$.

**35.** The internal bisector of the angle $A$ of a triangle $ABC$ cuts $BC$ in $P$ and the circum-circle of the triangle in $Q$. Prove that $AB \cdot AC = AP \cdot AQ$.

**36.** A sphere of radius 3 in. is placed with its centre ($C$) 7 in. in front of a screen, and a point source of light ($O$) is held 1 ft. in front of the screen, so that $OC$ is perpendicular to the screen. Find the radius of the circular shadow of the sphere thrown on to the screen.

# CHAPTER VII

## MISCELLANEOUS THEOREMS AND CONSTRUCTIONS

### Intersecting chords of a circle

**Theorem.**—If two chords of a circle intersect the product of the segments of one chord is equal to the product of the segments of the other.

Let $AB$, $CD$ be the two chords, intersecting in $P$. They may intersect inside the circle (as in Fig. 84 ($a$)) or outside the circle (as in Fig. 84 ($b$)).

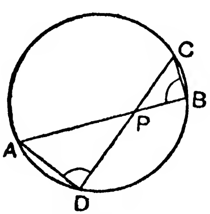



FIG. 84 ($a$).

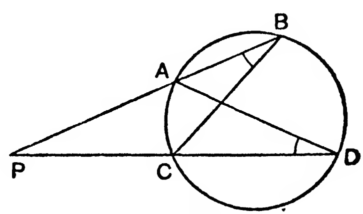



FIG. 84 ($b$).

$PA$, $PB$ are the "segments" of the chord $AB$; $PC$, $PD$ are the segments of the chord $CD$. Thus we have to prove that $PA \cdot PB = PC \cdot PD$.

Join $AD$ and $BC$.
Then in the $\triangle$'s $PAD$ and $PCB$,

$\angle PDA = \angle PBC$ (in the same segment, standing on arc $AC$).

$\angle APD = \angle CPB$ (vertically opposite angles in Fig. 84 (*a*), or same angle in Fig. 84 (*b*)).

Thus the triangles are equiangular and therefore similar.

Hence $$\frac{PA}{PC} = \frac{PD}{PB},$$

$$\therefore PA \,.\, PB = PC \,.\, PD.$$

**Corollary.**—The product of the segments of a chord drawn from a point outside a circle is equal to the square of the length of the tangent drawn from that point.

This is seen by swinging the chord $PCD$ round $P$, as shown in Fig. 85, until the points $C$ and $D$ coincide in the point $T$, when $PT$ will be a tangent to the circle. Thus

$$PA \,.\, PB = PC \,.\, PD = PC' \,.\, PD' = \;.\;.\;. = PT^2.$$

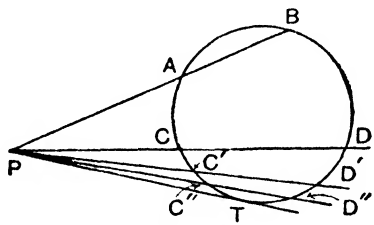



FIG. 85.

**Exercise.**—Prove the preceding corollary directly from Fig. 86. [Hint.—Prove the $\triangle$'s $PAT$, $PTB$ similar, using the "alternate segment theorem" (Part I, p. 209).]

The *converse* of the above theorem is: If $A$, $B$, $C$, $D$ are four points and $P$ is the intersection of $AB$ and $CD$ (produced if necessary), then if $PA \,.\, PB = PC \,.\, PD$ the four points are concyclic.

The proof is left as an exercise for the student.

[Hint.—A circle can be drawn to pass through three given points. Let the circle through $A$, $B$, $C$ cut $CD$ in $E$. Then prove that $D$ coincides with $E$.]

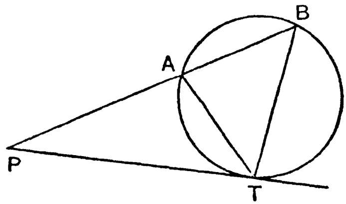



FIG. 86.

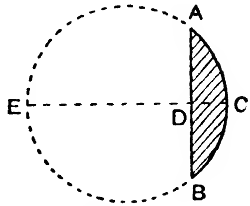



FIG. 87.

The *converse* of the corollary is: If $A$, $B$, $T$ are three points on a circle and $P$ is a point in $AB$ produced, then if $PT^2 = PA \cdot PB$ the line $PT$ is a tangent to the circle.

The proof is left to the student.

*Example.*—A plano-convex lens is of thickness $\frac{1}{2}$ in. and the diameter of its plane face is 2 in. Find the radius of its spherical surface.

Let $ACB$ represent a cross-section of the lens (Fig. 87), $D$ the centre of the plane face and $CE$ the diameter of the sphere through $D$. Then $AB = 2$ in.; $DC = \frac{1}{2}$ in.

If the radius of the sphere is $r$ in.,

$$ED = (2r - \tfrac{1}{2}) \text{ in.}$$

But

$$ED \cdot DC = AD \cdot DB$$

$$\therefore (2r - \tfrac{1}{2}) \times \tfrac{1}{2} = 1 \times 1$$

$$r - \tfrac{1}{4} = 1$$

$$r = 1\tfrac{1}{4}$$

Hence the radius of the spherical surface is $1\frac{1}{4}$ in.

## Length of chord of a circle

Let $AB$ be any chord of a circle, and $P$, $Q$ the mid-points of the minor and major arcs $AB$. Then $PQ$ is a diameter and it bisects $AB$ at right-angles at $N$ (Fig. 88).

Let the chord $AB = c$, $PN = h$ and the diameter $= d$. Then

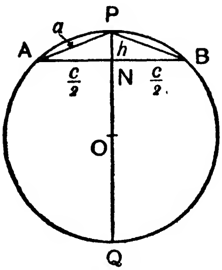



FIG. 88.

$$AN \,.\, NB = PN \,.\, NQ$$

$$\therefore \frac{c^2}{4} = h(d-h)$$

$$\therefore c^2 = 4h(d-h).$$

Let $AP = a$ ($AP$ is the "chord of half the arc"). Then

$$a^2 = h^2 + \left(\frac{c}{2}\right)^2 = h^2 + \frac{c^2}{4}$$

$$= h^2 + h(d-h) = h^2 + dh - h^2$$

$$\therefore a^2 = dh.$$

**Approximations to length of arc and area of segment**

The length of an arc and the area of a segment are evaluated exactly, in terms of the angle subtended at the centre, in Chapter IX.

The following approximations are sometimes useful:

$$\text{Length of minor arc } AB \simeq \frac{8a-c}{3},$$

where, as above, $a$ is the chord of half the arc and $c$ is the chord of the whole arc.

[This is known as Huygens' approximation.]

$$\text{Area of minor segment cut off by } AB \simeq \frac{h^3}{2c} + \tfrac{2}{3}ch.$$

Huygen's approximation is fairly accurate, the error never exceeding 1·2%. For small arcs the accuracy is much better than that.

The approximation for the area is not so good, though it never exceeds 4%. It should be used only for small segments.

**Mean proportional**

The mean proportional to two quantities $a$ and $b$ has been defined on p. 147 as the quantity $x$ such that $a : x = x : b$, i.e. such that $\frac{a}{x} = \frac{x}{b}$.

Hence $x^2 = ab$ and $x = \sqrt{ab}$.

[It is sometimes called the "geometric mean" of $a$ and $b$.]

**Construction.**—To draw the mean proportional to two given lengths.

Set out the two given lengths $AB$, $BC$ end to end in a straight line (Fig. 89).

Draw a semicircle on $AC$ as diameter.

Through $B$ draw the line $BD$ perpendicular to $AC$, and let it cut the semicircle in $D$.

Then $BD$ is the mean proportional to $AB$ and $BC$.

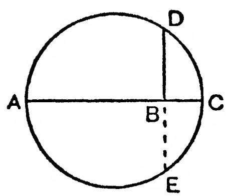



FIG. 89.

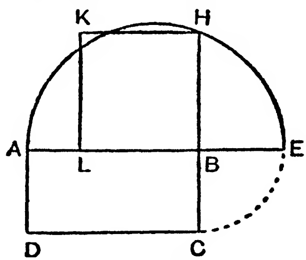



FIG. 90.

Proof.—Complete the circle on $AC$ as diameter and produce $DB$ to cut the circle in $E$.

$DB \,.\, BE = AB \,.\, BC$ (from the theorem on p. 151).

$$\text{But } BE = DB$$

$$\therefore DB^2 = AB \,.\, BC.$$

As an alternative proof we may notice that $\angle ADC$ is a right-angle (angle in semicircle) and then use Exercise XX, Question 15.

**Construction.**—To draw a square equal in area to a given rectangle.

Let $ABCD$ be the given rectangle (Fig. 90). Produce $AB$ to $E$, making $BE = BC$.

$$\text{Area of rectangle} = AB \,.\, BC$$
$$= AB \,.\, BE.$$

$\therefore$ Side of required square $= \sqrt{AB \,.\, BE} =$ mean proportional to $AB$ and $BE$. This is found by the previous construction. It is $BH$ in Fig. 90, and $BHKL$ is the required square.

**Construction.**—To draw the tangents to a circle from a given point outside it.

Let $P$ be the given point and $O$ the centre of the given circle. Draw the circle on $OP$ as diameter and let it cut the given circle in the points $A$ and $B$. Joint $PA$ and $PB$. Then $PA$ and $PB$ are the tangents from $P$ to the given circle.

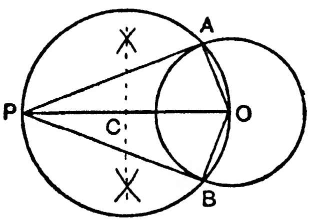


FIG. 91.

Proof.—Since the angle in a semicircle is a right angle, each of the angles $PAO$, $PBO$ is a right angle. Thus $PA$ is perpendicular to $OA$, and $PB$ perpendicular to $OB$.

But $OA$, $OB$ are radii of the given circle, and hence $PA$, $PB$ are tangents to that circle.

It is clear from symmetry that the two tangents from any point to a circle are equal in length.

This can be easily proved from Fig. 91. For the $\triangle$'s $PAO$, $PBO$ are each right-angled, have the common side $PO$, and have their sides $AO$, $BO$ equal. They are therefore congruent, and hence $PA = PB$.

### Common tangents of two circles

A line which touches each of two circles is called a common tangent to the two circles.

If two circles do not intersect they have four common tangents, as in Fig. 92 (*a*). *CD* and *EF* are called the *exterior* (or *direct*) common tangents; *HK* and *MN* are called the *interior* (or *transverse*) common tangents. If the two circles intersect they have only two common tangents, as in Fig. 92 (*b*).

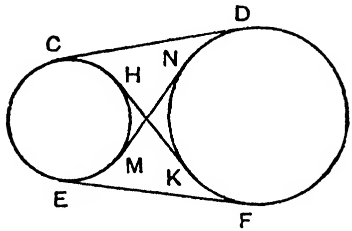



FIG. 92 (*a*).

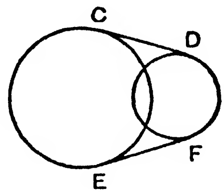



FIG. 92 (*b*).

When a belt passes over two pulleys the straight part of the belt is a common tangent to the two pulleys. In the case of an *open belt* (assuming there is no slackness) the tangents are exterior common tangents, and the two pulleys rotate in the same direction.

In the case of a *crossed belt* the tangents are interior common tangents, and the two pulleys rotate in opposite directions. It will be noticed from Fig. 92 (*a*) that the angle of lap is larger for a crossed belt than for an open one, and so there is less tendency to slip when the belt is crossed.

**Construction.**—To draw the common tangents to two circles.

(1) *Exterior common tangents.*

Let *A*, *B* be the centres of the two circles *CHE*, *DKF* (Fig. 93). Suppose that *B* is the centre of the larger circle.

With centre *B* draw a circle whose radius is equal to the *difference* of the radii of the given circles.

From *A* draw the tangents *AP*, *AQ* to that circle.

Join *BP* and *BQ*, and produce them to cut the larger circle in *D* and *F*.

Through $A$ draw $AC$ and $AE$ parallel to $BD$ and $BF$, and on the *same side* of $AB$ as those radii.

Join $CD$ and $EF$.

Then $CD$, $EF$ are the exterior common tangents.

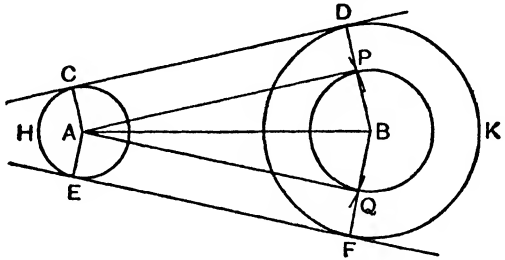



FIG. 93.

Proof.—Since $BP$ is equal to the difference of the radii $BD$ and $AC$, it follows that $PD = AC$. Thus $PD$ is equal and parallel to $AC$, and hence $APDC$ is a parallelogram (Part I, p. 183).

Further, since $AP$ is a tangent to the circle $PQ$, it is perpendicular to the radius $BP$ and hence to $PD$. Thus $\angle APD$ is a right angle, and so the parallelogram $APDC$ is a rectangle.

$CD$ is therefore perpendicular to $AC$ and to $BD$, and hence it is a tangent to each of the given circles. [The construction is the same whether the circles intersect or not.]

(2) *Interior Common Tangents.*

In this case with centre $B$ (the centre of the larger circle) draw a circle whose radius is equal to the *sum* of the radii of the given circles (Fig. 94).

From $A$ draw the tangents $AP$, $AQ$ to that circle.

Join $BP$ and $BQ$, and suppose they cut the larger circle in $D$ and $E$.

Through $A$ draw $AC$ parallel to $DB$ but on the *opposite side* of $AB$, and $AE$ parallel to $FB$ on the opposite side of $AB$. Join $CD$ and $EF$.

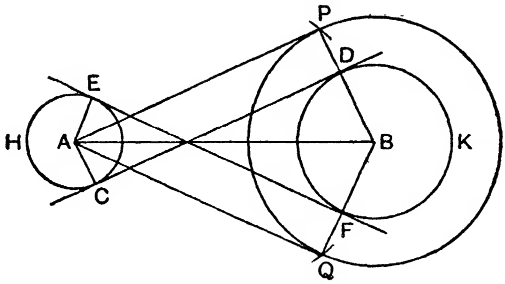



FIG. 94.

Thus $CD$, $EF$ are the interior common tangents.

The proof is left to the student.

**Medians of a triangle**

A line joining a vertex of a triangle to the mid-point of the opposite side is called a *median.*

A triangle has three medians, one through each vertex.

**Theorem.**—The three medians of a triangle meet in a point, which trisects each of them.

Let $Y$ be the mid-point of $AC$ and $Z$ the mid-point of $AB$, and let the medians $BY$, $CZ$ intersect in $G$.

Join $AG$, and let $AG$ produced cut $BC$ in $X$. Then we have to prove that $X$ is the mid-point of $BC$.

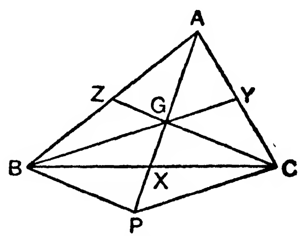



FIG. 95.

Draw $BP$ parallel to $ZC$, cutting $AG$ produced in $P$; join $CP$.

In $\triangle ABP$, $Z$ is the mid-point of $AB$ and $ZG$ is parallel to $BP$; hence $G$ is the mid-point of $AP$ (Theorem, p. 124).

In $\triangle APC$, $GY$ joins the mid-points of $AP$ and $AC$; hence $GY$ is parallel to $PC$, i.e. $BG$ is parallel to $PC$.

$BGCP$ is a parallelogram.

$X$ is the mid-point of $BC$. (Diagonals of a parallelogram bisect each other; Part I, p. 183.)

$\therefore AX$ is a median.

Hence the three medians meet in the point $G$.

Also $GX = \frac{1}{2}GP$. (Diagonals of a parallelogram bisect each other.)

But $GP = AG$ (since $G$ is mid-point of $AP$; proved above).

$\therefore GX = \frac{1}{2}AG$, i.e. $GX = \frac{1}{3}AX$.

Similarly it can be proved that $GY = \frac{1}{3}BY$ and $GZ = \frac{1}{3}CZ$.

Thus $G$ is the point of trisection of each of the medians.

[For another proof, depending on similar triangles, see Exercise XX, Question 19.]

**Centre of gravity, or centroid, of a triangular area**

In Fig. 96 the triangular area $ABC$ has been divided up into a number of thin strips by drawing lines parallel to the side $BC$. One such strip, $PQ$, is shown shaded. The centre of gravity of each strip, which may be regarded as a thin rod, is at its mid-point. Since the mid-points of all the strips lie on the median $AX$ (see Exercise XX, Question 34) it follows that the centre of gravity of the whole area lies on $AX$.

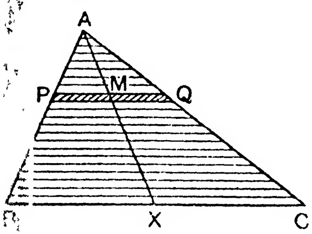


FIG. 96.

In the same way, by dividing up the triangle into thin strips parallel to $AB$ or $AC$, it is seen that the centre of gravity also lies on each of the other medians. It must therefore lie at the point of intersection of the medians.

Hence the centre of gravity of a triangular area is the point of intersection, $G$, of the medians. The point $G$ is usually called the *centroid* of the triangle.

and the inside width is 4 ft. 0 in. Find the outside

ct a line of length $\sqrt{6}$ in., by drawing the mean pro- in. and 3 in. Measure the length of your line, and h the value of $\sqrt{6}$ found from tables or by calculation.

$\sqrt{5}$ by a geometrical construction, and compare with given in your tables or found by calculation.

raw a rectangle of sides 2·6 in. and 1·5 in. Construct a e equal in area to the rectangle, and measure its side.

. Draw a circle of radius $1\frac{1}{4}$ in. and mark a point 4 in. from centre. Draw the tangents from the point to the circle, and asure their lengths.

Also find their lengths by calculation.

17. Draw two circles of radii 2 cm. and 3·5 cm. with their centres 12 cm. apart. Draw the exterior common tangents and measure their lengths.

18. Draw the two circles in Question 17, and draw the interior common tangents. Measure their lengths.

19. Calculate the lengths of the four common tangents to the two circles in Question 17.

20. Two circles of radii $a$ and $b$ have their centres distant $c$ apart, $c$ being greater than $a+b$. Calculate the lengths of (i) the exterior common tangents, (ii) the interior common tangents.

# CHAPTER VIII

## SOLID GEOMETRY

### Lines in space

If we draw any two lines in a plane (e.g. on a flat sheet of paper) they will intersect if produced far enough, provided they are not parallel.

Two lines in space, even if they are not parallel, i.e. not in the same direction, do not generally intersect. The student will see this fact if he takes a pencil in ea nd and holds them in any positions at random.

If two lines in space do not intersect nor [illegible] parallel, we say they are *skew* to each other. The following [illegible] of skew lines: (1) a line [illegible] ceiling of a room and a [illegible] parallel to the first one, o[illegible] floor, (2) the top edge of [illegible] cover of a book and the b[illegible] of the back cover, when the [illegible] partly open, (3) the diagonals $AC$ [illegible] $B'D'$ of opposite faces of the c[illegible] Fig. 104, (4) the lines $AC$ and $B$[illegible] Fig. 104.

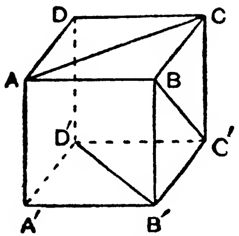


Fig. 104.

Since two lines which lie in the same plane either intersect or are parallel, two skew lines cannot lie in the same plane; that is, it is impossible to draw a plane to contain each of two skew lines.

## Planes in space

Any two planes in space intersect in a line, unless they are parallel.

We call that line the *line of intersection* of the planes.

A line in space cuts any plane in a point, unless it is parallel to the plane.

## Projection on a plane

If from any point outside a plane we draw the line perpendicular to the plane, the foot of the perpendicular is called the *projection* of that point on the plane.

If from every point of a curve, or figure of any sort in space, we draw perpendiculars on to a plane, the feet of the perpendiculars will lie on another curve or figure. This is called the projection of the original curve, or figure, on the plane.

[We sometimes speak of this projection as the *orthogonal projection* or *con*[illegible] to distinguish it from other types of projection [illegible] used in drawing.]

It is clear that the projection of a line on a plane is another line.

A *plan* is a projection on a horizontal plane; an *elevation* is a projection on a vertical plane. An object has a single plan, but may have various elevations according to the vertical plane on which it is projected; e.g. we speak of a side elevation, end elevation, etc.

### Angle between a line and a plane

If $M$, $N$ are the feet of the perpendiculars from $P$, $Q$ on the plane $ABCD$ (Fig. 105), the line $MN$ is the projection of the line $PQ$ on the plane.

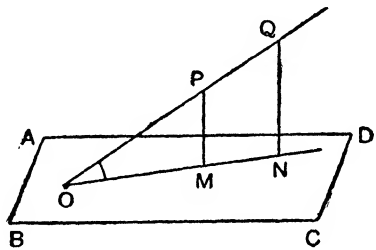



FIG. 105.

If the line $PQ$ cuts the plane in $O$, the line $MN$ passes through $O$.

*The angle between a line and a plane is defined as being the angle between the line and its projection on the plane.*

The angle between the line $PQ$ and the plane $ABCD$ in Fig. 105 is $\angle POM$.

### Angle between two planes

The angle between two planes, such as $ABCD$ and $ABEF$ in Fig. 106, is defined as follows: Take any point $P$ in their line of intersection $AB$, and draw the lines $PQ$ and $PR$, one in each of the planes, both lines being perpendicular to

*AB*. The angle between those lines, viz. $\angle QPR$, is defined to be the angle between the planes.

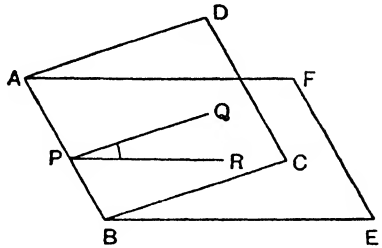



Fig. 106.

The student can make a model of Fig. 106 by drawing a line *AB* on a sheet of stiff paper, then drawing any line *QPR* perpendicular to *AB* on the paper, and finally folding the paper about *AB*.

**Angle between two skew lines**

The angle between two lines depends only on their directions.

The angle between two skew lines is therefore equal to the angle between two intersecting lines parallel to the given lines.

In practice the following is usually the most convenient way of defining the angle: Let *AB*, *CD* (Fig. 107) be two skew lines. Through any point *P* of *AB* draw *PQ* parallel

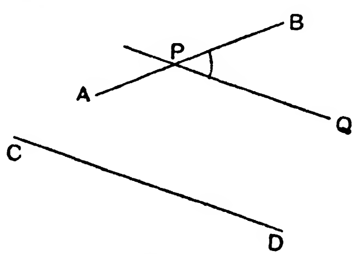



Fig. 107.

to $CD$. Then the angle between $AB$ and $CD$ is the angle between $AB$ and $PQ$, i.e. the $\angle BPQ$.

*Example.*—What is the angle between the diagonals $AC$ and $B'D'$ of opposite faces of the cube in Fig. 104 ?

$AC$ is clearly parallel to $A'C'$. Thus the angle between $AC$ and $B'D'$ is equal to the angle between $A'C'$ and $B'D'$, which is a right angle, since $A'B'C'D'$ is a square. Hence $AC$ and $B'D'$ are perpendicular to each other.

### Lines of greatest slope in a plane

If in Fig. 108 the plane $ABEF$ is horizontal, the lines of greatest slope (i.e. the steepest lines) in the plane $ABCD$ are those which are perpendicular to $AB$.

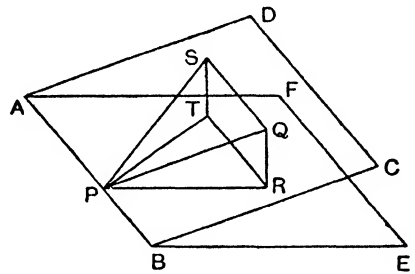



FIG. 108.

For let $PQ$ be the line through $P$ perpendicular to $AB$ in the plane $ABCD$, and let $PS$ be *any other* line through $P$ in that plane. Let $SQ$ be parallel to $AB$, and let $R$, $T$ be the projections of $Q$, $S$ on the horizontal plane. $SQ$ is horizontal, since it is parallel to $AB$ which is horizontal, and hence $RQ = TS$.

Also $PQ$ is perpendicular to $QS$, since $QS$ is parallel to $BA$, and hence $PQ$ is shorter than $PS$.

Thus in moving from $P$ to $Q$ we rise the same vertical distance as in moving from $P$ to $S$, but the distance travelled

in the first case is less than in the second. Hence $PQ$ is steeper than $PS$.

It is clear that the inclination of a plane to the horizontal is the same as the inclination of its lines of greatest slope.

### Direction of a line in space

Directions of lines on the ground, or in any horizontal plane, are given by compass-bearings, such as N.E. or S. 20° W. To indicate the direction of a line in space we need to specify two angles. For example, in describing the position of a star, we might say that it was seen in a direction N. 50° W. at an elevation of 42°. To locate the star from this information we should first turn our telescope through a *horizontal* angle of 50° from the North towards the West, and then raise it through an angle of 42° in the *vertical* plane. The angles are shown in Fig. 109; $NOA$ is a horizontal plane, $AOB$ is a vertical plane.

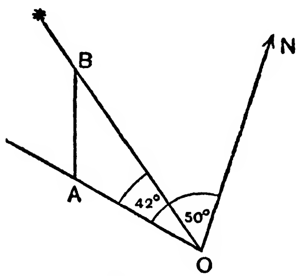


FIG. 109.

It is important to realize that angles measured by a compass are angles in a horizontal plane, so that when we say that a line, such as $OB$ in Fig. 109, has a bearing of N. 50° W. we mean that its projection on a horizontal plane is in the direction N. 50° W. Thus all lines in the same vertical plane have the same compass-bearing.

### Direction (or orientation) of a plane in space

A line which is perpendicular to a plane is called a ***normal*** to the plane. There is one normal through each point of a plane, and all the normals are parallel. Thus the direction of a plane (or its *orientation* as it is sometimes called) is known when the direction of its normals is known.

When we say that a wall (or, more correctly, one side of a wall) faces south, we mean that its normals (on that side) point south. If the wall is not vertical then its direction will be specified when we know, in addition, its inclination to the horizontal plane.

Thus if we say that a hill-side faces N. 70° E. at an inclination of 14°, we mean that it is inclined at 14° to the horizontal and that its normals bear N. 70° E.

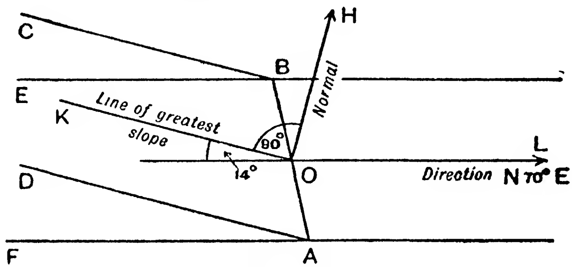



FIG. 110.

In Fig. 110, $ABCD$ is the plane of the hill-side, $ABEF$ a horizontal plane, $OH$ a normal and $OK$ the line of greatest slope through $O$. $KOHL$ is a vertical plane.

**Length of the projection of a line on a plane**

Let $PQ$ be a line of length $l$ making an angle $\theta$ with a plane $ABCD$ (Fig. 111), and let $pq$ be its projection on the plane.

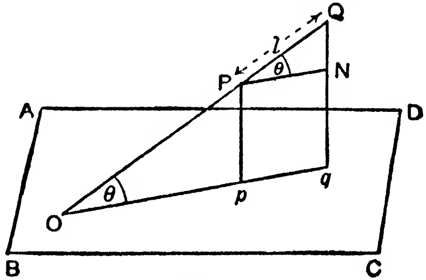



FIG. 111.

(The projections of points are often denoted by the corresponding small letters; thus here we have denoted the projection of the point $P$ by the letter $p$, and the projection of $Q$ by $q$.) Draw $PN$ parallel to $pq$. Then $\angle QPN = \angle POp$ (by parallels), and $\angle PNQ$ is a right angle.

Also $PNqp$ is a rectangle. $\therefore pq = PN$.

Now in the right-angled triangle $PNQ$,

$$\frac{PN}{PQ} = \cos\theta$$

$$\therefore PN = PQ\cos\theta = l\cos\theta$$

$$\therefore pq = l\cos\theta.$$

**Projection of an area**

In Fig. 112 $AB$ is the line of intersection of two planes $X$ and $Y$, and $PQRS$ is a rectangle in the plane $X$, whose sides are parallel to $AB$ and perpendicular to $AB$. The projection

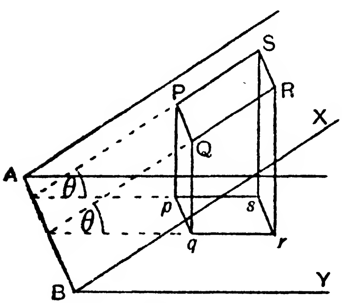



FIG. 112 (*a*).

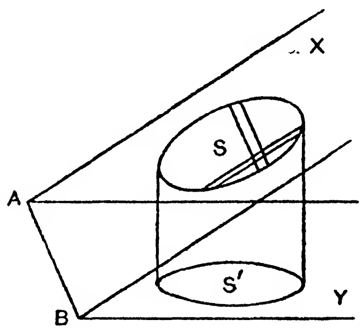



FIG. 112 (*b*).

of the rectangle on the plane $Y$ is $pqrs$. It is clear that $pq$ and $sr$ are parallel to $AB$, while $ps$ and $qr$ are perpendicular to $AB$; thus $pqrs$ also is a rectangle.

If the angle between the planes is $\theta$, then

$$pq = PQ,\ sr = SR,\ qr = QR\cos\theta,\ ps = PS\cos\theta.$$

$$\therefore \text{ Area } pqrs = pq \times qr = PQ \times QR \cos\theta$$
$$= (\text{area } PQRS) \times \cos\theta.$$

Now suppose we consider *any* area $S$ in the plane $X$ (Fig. 112 ($b$)) and its projection $S'$ on the plane $Y$.

As explained in Chapter VI (p. 142), we can regard the area $S$ as built up of rectangles whose sides are parallel to $AB$ and perpendicular to $AB$. When we project on to the plane $Y$, the area of each of these rectangles is reduced in the ratio $\cos\theta : 1$, and hence the sum of their areas is reduced in the same ratio. Thus

$$S' = S\cos\theta.$$

## Reduction of lengths in projection

If, in Fig. 112, we regard $Y$ as a horizontal plane, the lines of greatest slope in the plane $X$ are the lines perpendicular to $AB$. Any line, in the plane $X$, oblique to $AB$ is inclined to the plane $Y$ at an angle less than $\theta$, and the cosine of its angle of inclination is therefore greater than $\cos\theta$. Hence for such lines the ratio *length of projection : original length* is greater than $\cos\theta$, the ratio depending on the inclination of the line.

The lines of greatest slope, therefore, suffer the greatest reduction in length.

Thus, in projecting from one plane on to another, all lines parallel to the line of intersection of the planes are unaltered in length, all lines perpendicular to the line of intersection are reduced in the ratio $\cos\theta : 1$, and all other lines are reduced in some intermediate ratio.

## The ellipse

If a circle in a plane $X$ (Fig. 113) is projected on to another plane $Y$, the oval-shaped figure obtained is called an *ellipse.*

The centre of the circle, $H$, projects into a point $O$ which is called the *centre* of the ellipse. Chords of the ellipse which pass through its centre are called *diameters* of the ellipse. Diameters of the circle project into diameters of the ellipse.

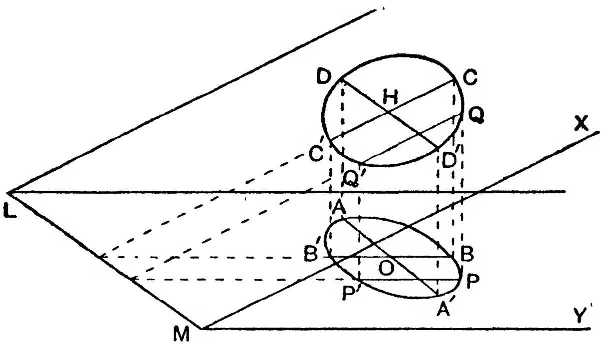



FIG. 113.

The diameter $DD'$ parallel to $LM$, the line of intersection of the planes, is unaltered in length by projection ; the diameter $CC'$ perpendicular to $LM$ is reduced more than any other diameter. Hence their projections, $AA'$ and $BB'$, are the greatest and least diameters of the ellipse ; we call $AA'$ the *major axis* and $BB'$ the *minor axis* of the ellipse.

If the radius of the circle is $a$, and the angle between the planes is $\theta$, $AA' = 2a$ and $BB' = 2a \cos \theta$.

Also, any chord $QQ'$ of the circle which is perpendicular to $LM$ projects into a chord $PP'$ of the ellipse, such that $PP' = QQ' \cos \theta$.

In Fig. 114 the circle and the ellipse are drawn in the same plane with the diameters $DD'$ and $AA'$ superposed.*

The lengths of the major and minor axes of an ellipse are usually denoted by $2a$ and $2b$ respectively. Thus

$$2b = BB' = 2a \cos \theta$$

$$\therefore \cos \theta = \frac{b}{a}.$$

* We may regard Fig. 114 as obtained by projecting the circle $ACA'C'A$ on a plane through $AA'$ inclined to the plane of the circle, and then rotating the second plane together with the projected figure (the ellipse) back into the plane of the circle.

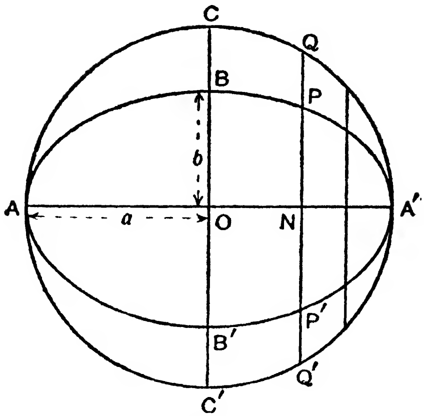



FIG. 114.

Hence an ellipse of major axis $2a$ and minor axis $2b$ may be regarded as obtained by projecting a circle of radius $a$ on a plane inclined to the plane of the circle at an angle $\theta$ such that $\cos\theta = \frac{b}{a}$.

Since $PP' = QQ' \cos\theta$ and $N$ is the mid-point of both $QQ'$ and $PP'$,

$$NP = NQ \cos\theta = \frac{b}{a} \times NQ.$$

Hence we have the following construction for drawing an ellipse with axes of lengths $2a$, $2b$:

Draw a circle of radius $a$, and any diameter $AA'$. From any point $Q$ of the circle draw $QN$ perpendicular to $AA'$ and mark the point $P$ on $QN$ such that $NP = \frac{b}{a} \times NQ$. Do this for a large number of positions of $Q$, and join up, by a smooth curve, all the points such as $P$ obtained in this way. The resulting curve will be the required ellipse.

The way in which this construction leads to the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has been shown on page 65.

There are various simple mechanical constructions for drawing an ellipse by means of a *trammel.* For these the student is referred to any book on geometrical drawing.

**Area of an ellipse**

$$\begin{aligned}\text{Area of ellipse} &= (\text{area of circle}) \times \cos\theta \\ &= \pi a^2 \times \frac{b}{a} \\ &= \pi ab.\end{aligned}$$

*Example.*—Find the area of a section of a circular cylinder of diameter $d$ by a plane inclined at an angle $\alpha$ to the axis of the cylinder.

Fig. 115 shows the section, $PQ$, and also a cross-section, $MN$ (i.e. a section by a plane perpendicular to the axis).

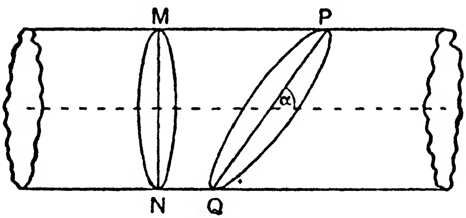



FIG. 115.

The section $MN$ is the orthogonal projection of the section $PQ$ on the plane $MN$.

The angle between the two planes is $90° - \alpha$.

$$\therefore \text{Area of section } MN = \text{area of section } PQ \times \cos(90° - \alpha)$$
$$= \text{area of section } PQ \times \sin\alpha.$$

$$\therefore \text{Area of section } PQ = \frac{\text{area of section } MN}{\sin\alpha}$$
$$= \frac{\pi d^2}{4\sin\alpha}.$$

**Latitude and longitude**

The Earth is very nearly a sphere and for most practical purposes may be regarded as such. Its radius is approximately 3960 miles.

If, in Fig. 116, $P$ is any point on the Earth's surface and $N$, $S$ are the North and South poles, the plane $NPS$ cuts the sphere in a circle. The semicircle $NPS$, i.e. any semicircle on $NS$ as diameter, is called a *meridian*. The angle between the plane of this meridian and the plane of the meridian

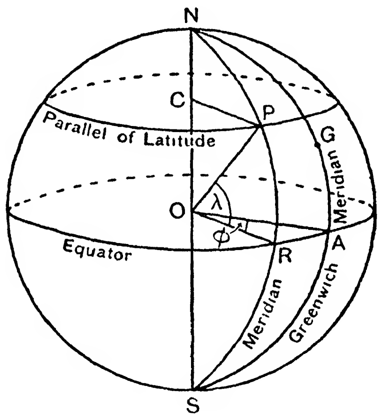



Fig. 116.

through Greenwich (the *Greenwich meridian*) is called the *longitude* of $P$. Longitude is reckoned East or West of Greenwich, up to 180° in each direction. All points on the same meridian have the same longitude, of course. In Fig. 116 the longitude of $P$ is the angle $\phi$ (West).

The plane through $P$ parallel to the plane of the Equator, i.e. perpendicular to the axis $NS$, cuts the sphere in a circle which is called a circle of latitude or, more generally, a *parallel of latitude*. The angle between $OP$ and the plane of the Equator is called the *latitude* of $P$. Latitude is reckoned

north or south of the Equator, up to 90° in each direction. All points on the same parallel of latitude have the same latitude, of course. In Fig. 116 the latitude of $P$ is $\lambda$ (North).

The position of a point on the Earth's surface is determined by its latitude and longitude (usually abbreviated Lat. and Long.); for example New York is in Lat. 40° 20′ N., Long. 74° 0′ W.

## Exercise XXII

**1.** A rectangular brick measures 9 in. × $4\frac{1}{2}$ in. × 3 in. What is the distance between two opposite corners?

**2.** If each side of the cube in Fig. 117 is of length $a$, find (i) the length of the diagonal $AC'$, (ii) the angle between $AC'$ and the face $A'B'C'D'$, (iii) the angle between $AC'$ and $AD$.

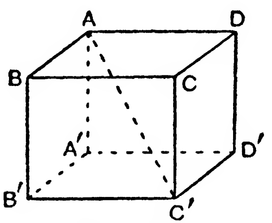


Fig. 117.

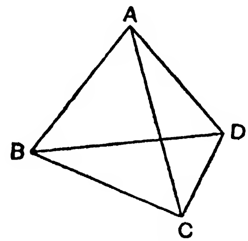


Fig. 118.

**3.** A vertical mast is strengthened by three stay-wires, each 35 ft. long, attached to a point of the mast 28 ft. above the ground, the wires being arranged symmetrically around the mast. Find (i) the angle each wire makes with the ground, (ii) the distance of the foot of each wire from the mast, (iii) the distance between the feet of the wires. [Assume the wires are straight and neglect the thickness of the mast.]

**4.** Fig. 118 shows a *regular tetrahedron* (i.e. a triangular pyramid whose six edges are all equal, so that all its faces are equilateral triangles). Find (i) the angle between the edge $AB$ and the face $BCD$, (ii) the angle between any two faces (e.g. $ABC$ and $BCD$).

**5.** Prove that the figure obtained by joining the mid-points of consecutive sides of a skew quadrilateral (i.e. a quadrilateral not in one plane) is a parallelogram.

6. A board in the shape of an equilateral triangle is placed in the corner of a room, with one edge resting on the floor and one against each of the walls. Find the inclination of the board to the floor.

7. A door, 6 ft. 6 in. high and 2 ft. 6 in. wide, originally shut, is opened through an angle of 30°. Find the angle between a diagonal of the door and the wall.

8. A 60° set square is placed on a table and is then rotated about its longest side through an angle of 45°. What are then the inclinations of the other two sides to the table ?

9. A pyramid has a square base of side 6 in. and its vertical height is 8 in. Find (i) the angle between a sloping edge and the base, (ii) the angle between a sloping face and the base.

10. Fig. 119 shows a *hipped roof*. The *ridge* $AB$ is horizontal, and so is the plane $CDEF$ (the plane of the *eaves*). $AC$, $AF$, $BD$, $BE$ are *the hip rafters*. If $AB = 15$ ft., $CD = FE = 20$ ft., $DE = CF = 14$ ft. ($CDEF$ being a rectangle), and the ridge is 5 ft. (vertically)

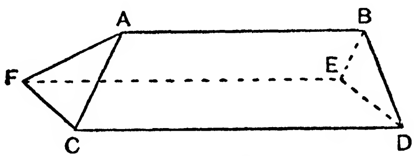



Fig. 119.

above the plane of the eaves, find (i) the area of the roof, (ii) the length of the hip rafters, (iii) the slope (i.e. inclination to the horizontal) of the hip rafters, (iv) the slopes of the roof-faces. [Hint.—Draw the perpendicular from $A$ to the plane of the eaves.]

11. The ridges of two sloping roofs meet at right angles ; if the roofs are both inclined at 45° to the horizontal, find the inclination of their line of intersection.

[Hint.—Draw a plan of the roofs.]

12. A line of greatest slope on a hill-side has a gradient of 1 (vertical) in 10 (up the plane). What is the gradient of a road which makes an angle of 20° with the lines of greatest slope ?

13. A hill-side, which may be regarded as a plane, faces due south and is inclined at 20° to the horizontal. Find the inclination of a road on the hill-side in the direction N.E.

**14.** A prism whose cross-section is an equilateral triangle of side 4 in. is cut by a plane inclined at 60° to the edges of the prism. Find the area of the section.

**15.** A circular disc of diameter 1 ft. is held at right-angles to the sun's rays when the sun's altitude (i.e. elevation) is 55°. What is the area of the shadow on the ground ?

**16.** Fig. 120 shows the side-elevation and front-elevation of a semicircular dormer window. Find the area cut away from the main roof.

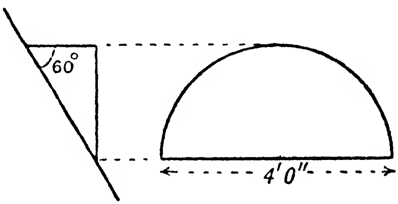



FIG. 120.

**17.** Find the diameter of a circle whose area is equal to that of an ellipse of axes 10 in. and 6 in.

**18.** A nautical mile is the length of arc on the Equator which subtends an angle of 1 minute at the centre of the Earth. Find the length of a nautical mile in feet.

**19.** Calculate the distance of Greenwich (Lat. 51° 29′ N.) from (i) the Equator, (ii) the North Pole.

**20.** Find the length of the Arctic Circle (parallel of Latitude $66\frac{1}{2}°$).

## Mensuration. Volumes and surfaces

We shall assume the formulæ for the volumes and surfaces of the following simple solids.

Some of these formulæ have been demonstrated in Part I, Chap. XV ; the remainder are proved in Chap. XVI of this Part or in Part III.

**Prism.**—Volume = (area of base) × (perpendicular distance between parallel ends).

**Cylinder.**—This may be regarded as a special case of a prism. For a circular cylinder, if the radius of the base $=r$, and height $=h$, then area of base $=\pi r^2$.

$$\therefore \text{ volume} = \pi r^2 h.$$

Area of curved surface = (circumference of base) × height

$$= 2\pi rh.$$

**Pyramid.** — Volume $= \frac{1}{3}$(area of base) × (perpendicular height).

**Cone.**—This may be regarded as a special case of a pyramid. If the radius of the base $=r$, and height $=h$, area of base $=\pi r^2$.

$$\therefore \text{ volume} = \tfrac{1}{3}\pi r^2 h.$$

If $l$ is the length of the slant height,

area of curved surface $= \frac{1}{2}$ (circumference of base) × (slant height)

$$= \tfrac{1}{2} \times 2\pi r \times l$$

$$= \pi rl.$$

**Sphere.**—If the radius $=r$,

$$\text{volume} = \tfrac{4}{3}\pi r^3;$$

$$\text{surface area} = 4\pi r^2.$$

### Volumes of similar solids

Similar solids have been defined in Chap. VI as solids which are of the same shape but different in size.

Of two similar solids, one may be regarded as an enlargement of the other.

One of the simplest solids is a rectangular block, shaped like a brick. It is sometimes called a *cuboid*.

Fig. 121 shows two similar rectangular blocks. If the edges of one of the blocks are of lengths $a$, $b$, $c$, the corresponding edges of the other block are equal multiples of those lengths; say $na$, $nb$, $nc$.

The volume of the first block is $a \times b \times c$, i.e. $abc$; the volume of the second block is $na \times nb \times nc$, i.e. $n^3abc$. Hence the

ratio of the volumes of the blocks is $\dfrac{n^3abc}{abc} = \dfrac{n^3}{1}$. Thus the ratio of the volumes of similar rectangular blocks is equal to the *cube* of the ratio of corresponding edges.

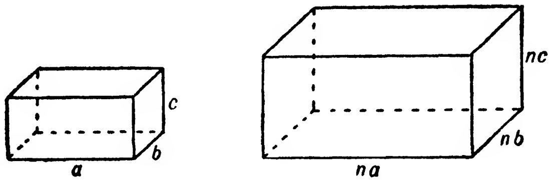



FIG. 121.

We have already seen (p. 142) how any plane area may be regarded as built up of rectangles. In much the same way any solid, whatever its shape, may be regarded as built up of rectangular blocks.

If the solid is enlarged, or reduced, every line in it is enlarged, or reduced, in the same ratio. If this ratio is $n : 1$, the volume of each of the inscribed rectangular blocks is increased in the ratio $n^3 : 1$, and hence the sum of the volumes of the blocks also is increased in the ratio $n^3 : 1$. Thus the ratio of the volumes of two similar solids is equal to the cube of the ratio of the lengths of corresponding lines in the two solids.

We usually express this in the form :

*The volumes of similar solids are proportional to the cubes of their corresponding linear dimensions.*

Thus, for example, if the linear dimensions of a body are doubled its volume is increased eight-fold.

*Example.*—Two iron castings are made to similar patterns, one being 3 ft. 6 in. long, the other 5 ft. 4 in. long. If the smaller casting weighs 14 cwt., what is the weight of the larger one ?

Since the castings are of the same material, their weights are proportional to their volumes.

$$\frac{\text{Volume of larger casting}}{\text{Volume of smaller casting}} = \left(\frac{5 \text{ ft. } 4 \text{ in.}}{3 \text{ ft. } 6 \text{ in.}}\right)^3$$
$$= \left(\frac{64 \text{ in.}}{42 \text{ in.}}\right)^3$$
$$= \left(\tfrac{32}{21}\right)^3 = (1{\cdot}524)^3 \simeq 3{\cdot}54.$$

$$\therefore \frac{\text{Wt. of larger casting}}{\text{Wt. of smaller casting}} = 3{\cdot}54$$

$$\therefore \text{Wt. of larger casting} = 3{\cdot}54 \times 14 \text{ cwt.}$$
$$= 49{\cdot}56 \text{ cwt.}$$
$$\simeq 2 \text{ tons } 9\tfrac{1}{2} \text{ cwt.}$$

**Areas of surfaces of similar solids**

If a solid bounded by plane faces is enlarged so that every length in it is doubled, the area of each face is increased fourfold and hence the total surface area of the solid also is increased fourfold.

More generally, if each length is increased or decreased in the ratio $n : 1$, the total surface area increases, or decreases, in the ratio $n^2 : 1$.

This is true also for solids bounded by curved surfaces; that is, *the surface-areas of similar solids are proportional to the squares of their corresponding linear dimensions.*

As an example, we might verify this for the case of a cylinder by using the formula for its surface area. If the radius of the cylinder is $r$ and its height $h$,

$$\text{area of curved surface} = 2\pi rh,$$
$$\text{area of each end} = \pi r^2.$$
$$\therefore \text{Total surface area} = 2\pi rh + 2\pi r^2.$$

For a similar cylinder, whose radius and height are $nr$ and $nh$,

$$\text{total surface area} = 2\pi nr \,.\, nh + 2\pi(nr)^2$$
$$= 2\pi n^2 rh + 2\pi n^2 r^2$$
$$= n^2(2\pi rh + 2\pi r^2).$$

Hence the ratio of their surface areas is $n^2 : 1$.

*Example.*—A manufacturer makes three types of motor-car headlamps, all of the same pattern but of different sizes. The diameters of the glasses for the three types are 12 in., 8 in., 6 in. If it costs 5 shillings to chromium plate the largest size, how much does it cost for the other sizes? (Assume the same thickness of plating in each case.)

The areas of the surfaces to be plated are proportional to the squares of corresponding linear dimensions, i.e. are in the ratios $12^2 : 8^2 : 6^2$, i.e. $144 : 64 : 36$, i.e. $36 : 16 : 9$.

Hence the cost of plating the second size lamp

$$=\tfrac{16}{36} \times 5 \text{ shillings} = \tfrac{20}{9} \text{ shillings} \simeq 2s.\ 2\tfrac{1}{2}d.,$$

and the cost of plating the smallest size lamp

$$=\tfrac{9}{36} \times 5 \text{ shillings} = \tfrac{5}{4} \text{ shillings} = 1s.\ 3d.$$

**Frustum of a pyramid**

By a *frustum* of a pyramid is meant a portion of a pyramid cut off between two planes parallel to the base. In Fig. 122

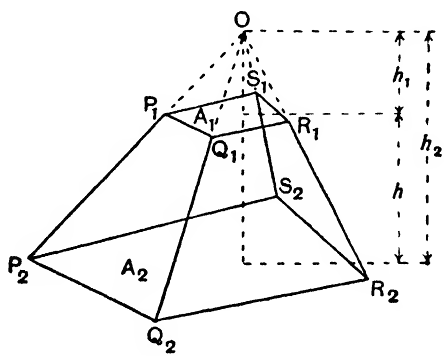



FIG. 122.

the solid between the planes $P_1Q_1R_1S_1$, and $P_2Q_2R_2S_2$ is a frustum of a pyramid.

Volume of frustum = volume of pyramid $OP_2Q_2R_2S_2$ − volume of pyramid $OP_1Q_1R_1S_1$.

Let the volumes of the pyramids $OP_1Q_1R_1S_1$, $OP_2Q_2R_2S_2$ be $V_1$, $V_2$ and their perpendicular heights $h_1$, $h_2$, and the areas of their bases $A_1$, $A_2$.

Let the volume of the frustum be $V$ and the perpendicular distance between its parallel ends be $h$.

Then $h = h_2 - h_1$, and $V = V_2 - V_1$.

The pyramids $OP_1Q_1R_1S_1$, $OP_2Q_2R_2S_2$ are similar solids, and hence $\dfrac{V_1}{V_2} = \left(\dfrac{h_1}{h_2}\right)^3$.

$$\therefore\ V = V_2 - V_1 = V_2\left(1 - \frac{V_1}{V_2}\right) = V_2\left(1 - \frac{h_1^3}{h_2^3}\right)$$

$$= \frac{V_2}{h_2^3}(h_2^3 - h_1^3) = \frac{V_2}{h_2^3}(h_2 - h_1)(h_2^2 + h_2h_1 + h_1^2)$$

$$= \tfrac{1}{3}\frac{A_2h_2}{h_2^3}\,.\,h(h_1^2 + h_1h_2 + h_2^2)$$

$$= \tfrac{1}{3}A_2h\left(\frac{h_1^2}{h_2^2} + \frac{h_1}{h_2} + 1\right).$$

The bases $P_1Q_1R_1S_1$, $P_2Q_2R_2S_2$ are similar figures, since they are in parallel planes and in perspective (from $O$). Hence

$$\frac{A_1}{A_2} = \left(\frac{h_1}{h_2}\right)^2$$

$$\therefore\ V = \tfrac{1}{3}A_2h\left(\frac{A_1}{A_2} + \sqrt{\frac{A_1}{A_2}} + 1\right)$$

$$= \tfrac{1}{3}h(A_1 + \sqrt{A_1A_2} + A_2).$$

**Frustum of a cone.**

Since a circle may be regarded as the limit of an inscribed polygon when the number of sides increases indefinitely, a cone may be regarded as the limiting case of a pyramid when the base becomes a circle.

Thus the previous formula is true also for a frustum of a cone, shown in Fig. 123. It may be expressed in another form as follows.

If the radii of the circular ends are $r_1$ and $r_2$, $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$.

$$\therefore\ V = \text{vol. of frustum of cone} = \tfrac{1}{3}h(\pi r_1^2 + \sqrt{\pi^2 r_1^2 r_2^2} + \pi r_2^2)$$
$$= \tfrac{1}{3}\pi h(r_1^2 + r_1 r_2 + r_2^2).$$

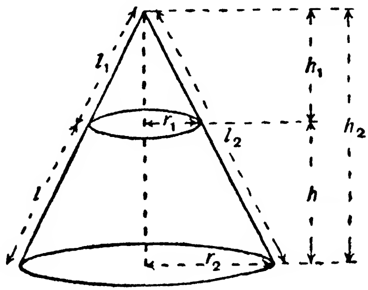



FIG. 123.

This result may be obtained also by using the formula $\frac{1}{3}\pi r^2 h$ for the volume of a cone, treating the frustum as the difference of the two cones shown in the figure.

The area of the curved surface of the frustum, which we shall denote by $S$, is the difference of the areas of the curved surfaces of the two cones shown in the figure.

Denoting the slant heights of the cones by $l_1$, $l_2$ and the slant height of the frustum by $l$, so that $l = l_2 - l_1$,

$$S = \pi r_2 l_2 - \pi r_1 l_1 = \pi r_2 l_2\left(1 - \frac{r_1}{r_2}\frac{l_1}{l_2}\right).$$

But, from similar triangles, $\dfrac{r_1}{r_2} = \dfrac{l_1}{l_2}$

$$\therefore\ S = \pi r_2 l_2\left(1 - \frac{l_1^2}{l_2^2}\right)$$
$$= \pi r_2 l_2\left(\frac{l_2^2 - l_1^2}{l_2^2}\right) = \pi\frac{r_2}{l_2}(l_2 - l_1)(l_2 + l_1)$$

$$=\pi r_2 l\left(1+\frac{l_1}{l_2}\right)=\pi r_2 l\left(1+\frac{r_1}{r_2}\right)$$
$$=\pi(r_1+r_2)l.$$

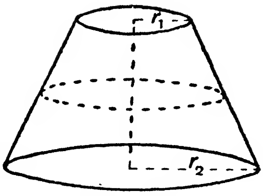



FIG. 124.

The radius of the mid-section of the frustum, i.e. the section by the plane mid-way between the ends and parallel to them (shown by the dotted circle in Fig. 124) is $\frac{1}{2}(r_1+r_2)$. Its circumference $=2\pi \cdot \frac{1}{2}(r_1+r_2)=\pi(r_1+r_2)$.

Hence we can express the surface area of a frustum of a cone in the form

$$S=(\text{circumference of mid-section})\times(\text{slant height}).$$

## Zone of a sphere

A *zone* of a sphere is a part cut off between two parallel planes.

Its plane faces are circles. If their radii are $r_1$, $r_2$ and the distance between the parallel planes is $h$ (Fig. 125), it can be proved (see Part III) that

$$\text{volume of zone}=\tfrac{1}{6}\pi h\{3(r_1^2+r_2^2)+h^2\}.$$

If the radius of the sphere is $r$, it can be proved (see Part III) that

$$\text{area of curved surface of zone}=2\pi rh.$$

This is equal to the area cut off between the same planes on the cylinder which circumscribes the sphere and whose axis is perpendicular to the planes.

It should be noted that this area depends only on the distance between the planes and not on the positions of the planes.

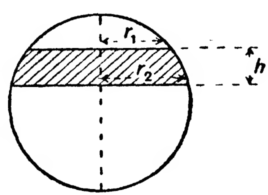



FIG. 125.

**Segment (or cap) of a sphere**

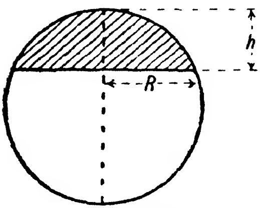



FIG. 126.

A *segment*, or *cap*, of a sphere is a part cut off by a plane (Fig. 126). It is a special case of a zone when one of the planes touches the sphere.

If the radius of the plane face is $R$ and the thickness of the cap is $h$, the volume is obtained from the formula for the volume of a zone by putting $r_1 = 0$, $r_2 = R$.

$\therefore$ Volume of segment $= \frac{1}{6}\pi h(3R^2 + h^2)$.

Area of curved surface of segment $= 2\pi rh$.

**Oblique frustum of a cylinder**

If a cylinder is cut by a plane $AB$ (Fig. 127) not parallel to the end faces, the volume $ABDC$ cut off is clearly equal to the volume of the cylinder $MNDC$, since vol. $BPN$ = vol. $APM$.

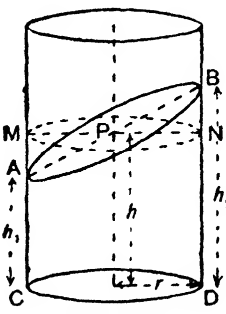



FIG. 127.

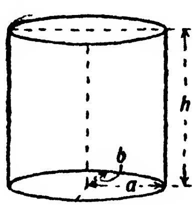



FIG. 128.

$\therefore$ Vol. of oblique frustum $ABDC = \pi r^2 h$.

If $h_1$, $h_2$ are the lengths of the shortest and longest generators of the frustum, $h = \frac{1}{2}(h_1 + h_2)$.

$\therefore$ Vol. of frustum $= \pi r^2 \left(\dfrac{h_1 + h_2}{2}\right)$.

Also, area of curved surface of frustum = area of cylinder $MNDC = 2\pi rh = \pi r(h_1 + h_2)$.

### Elliptic cylinder

If the cross-section of a cylinder is an ellipse whose principal axes are of lengths $2a$, $2b$,

volume of cylinder = (area of base) × height $= \pi ab.\ h$.

## Exercise XXIII

1. An iron sphere of diameter 8 in. weighs 70 lb. What is the weight of one of diameter 6 in.?

2. A set of weights for scales are similar solids; the 4 oz. weight is $\frac{1}{2}$ in. thick. What are the thicknesses of the 1 oz., 8 oz. and 1 lb. weights?

3. The surface area of one sphere is eight times that of another; find the ratio of the radii and the ratio of their volumes.

4. Two equal spheres each of radius 5 cm. are melted down into a single sphere. What is its radius?

5. Two similar lead cylinders, one 3 in. long, the other 5 in. long, are melted down and made into a single similar cylinder. What is its length?

6. A pyramid 10 in. high is cut into two parts by a plane parallel to the base and 6 in. from the base. What is the ratio of the volumes of the two parts?

7. A cone of height 6 cm. is to be divided into two parts of equal weights by a plane parallel to the base. Find the distance of the plane from the base.

8. Find the volume of a right prism 1 ft. 6 in. long whose cross-section is an equilateral triangle of side 3 in.

9. A cubic foot of copper is drawn into a wire 800 yards long. What is the diameter of the wire?

10. A mound of earth is in the shape of a pyramid 2 ft. 6 in. high on a square base of side 3 ft. 3 in. Find the volume of earth in the mound.

11. Find the volume and surface-area of a regular tetrahedron of side 8 cm.

12. A sphere of 2 in. diameter is beaten out into a circular sheet 0·015 in. thick. Find the radius of the sheet.

**13.** A hollow cast-iron sphere weighs 240 lb. and its external diameter is 14 in. Find the thickness of the iron. [1 cu. in. of cast iron weighs 0·26 lb.]

**14.** Find the weight per foot length of lead pipe of $1\frac{1}{2}$ in. bore, $\frac{3}{16}$ in. thick. [1 cu. ft. of lead weighs 711 lb.]

**15.** A glass tube 15 cm. long, of external diameter 4 mm. weighs 4 gm. Find the inside diameter. [Sp. gr. of glass = 2·52.]

**16.** Find the volume of a spherical shell of internal radius $r$ and thickness $t$, and show that if $t$ is very small compared with $r$ the volume is approximately $4\pi r^2 t$.

**17.** A reservoir containing a million gallons of water is emptied by a pipe of diameter 2 ft. 4 in. in 3 hours. Find the rate of flow in the pipe. [$6\frac{1}{4}$ gall. = 1 cu. ft.]

**18.** A pipe of diameter 3 ft. is to be replaced by two pipes, one having twice the diameter of the other, which shall together carry the same volume of water as the single pipe. Find their diameters.

**19.** Find the volume of the air-space under the hipped roof in Exercise XXII, Question 10 (p. 179).

[Hint.—Draw vertical planes through $A$ and $B$ so as divide the volume into a prism and two pyramids.]

**20.** A swimming bath is 110 ft. long and 25 ft. wide. The water is 3 ft. deep at one end and 6 ft. 6 in. at the other. Find the volume of the water.

**21.** Find the volume of a log of timber, 6 ft. long, which tapers from 8 in. square at one end to 3 in. square at the other.

**22.** A friction clutch is in the shape of a frustum of a cone of end diameters 5 in. and 7 in., the distance between the plane ends being 4 in. Find the area of the curved surface.

**23.** Fig. 129 shows a vertical cross-section of the head of a bolt, horizontal sections being circles. Find its volume.

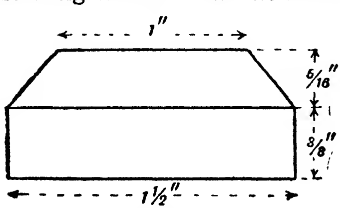



FIG. 129.

24. A hollow shaft 3 ft. 6 in. long is of uniform thickness throughout, and has a taper of 1 in 20. At the narrow end the external diameter is 4 in. and the internal diameter 3 in. Find the volume of metal in the shaft.

25. A glass ash-tray is in the form of a frustum of a cone with a spherical segment scooped out from the top, the dimensions being as shown in Fig. 130. Find the volume of glass.

26. The vertical cross-section of a casting is shown in 131, the bottom face being circular. Find the volume of metal in the casting.

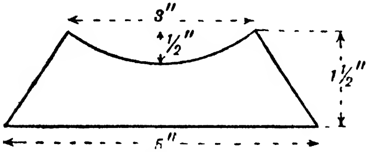



FIG. 130.

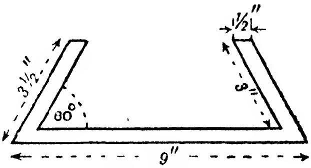



FIG. 131.

27. Find the *total* surface area of the frustum of a cylinder shown in Fig. 132.

28. Find the ratios of the surface areas of a cube, sphere and regular tetrahedron of equal volume.

29. A fine wire of diameter $d$ is wound (close-coiled) on the outside of a cylinder of length $l$ and diameter $D$. Find the length of the wire.

30. A right-angled triangle whose sides are 3 cm., 4 cm., 5 cm. revolves about its hypotenuse. Find the volume of the double cone generated.

31. Fig. 133 shows a section of a collar for a shaft. Find its volume.

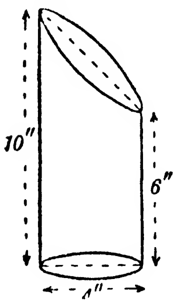



FIG. 132.

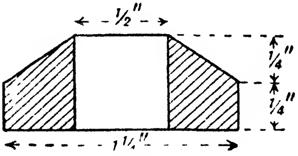



FIG. 133.

32. The largest possible cone of vertical angle 60° is cut out from a sphere. Find the ratio of its volume to that of the sphere.

33. An elbow-joint (Fig. 134) consists of two pieces of metal piping at right-angles. The external diameter of each pipe is 3 in. and the metal is $\frac{1}{8}$ in. thick. Find the volume of the metal.

34. Find the weight of the steel rivet shown in cross-section in Fig. 135, the total length of the rivet being $1\frac{5}{8}''$. [1 cu. in. of steel weighs 0·29 lb.]

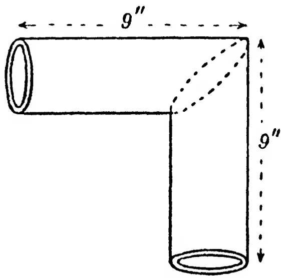



FIG. 134.

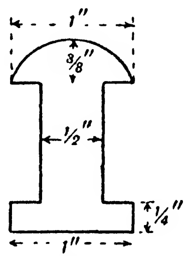



FIG. 135.

35. Prove that the volume of a segment of a sphere is $\frac{1}{3}\pi h^2(3r-h)$, where $r$ is the radius of the sphere and $h$ is the height of the segment.

36. If $S$ is the area of the curved surface of a frustum of a cone, $r_1$ and $r_2$ the radii of its ends, $l$ the slant height and $\theta$ the angle between the slant height and the base, prove by projection that $l \cos\theta = r_2 - r_1$ and $S \cos\theta = \pi(r_2^2 - r_1^2)$; deduce that $S = \pi(r_1 + r_2)l$.

37. Obtain the formula (p. 187) for the area of the curved surface of a frustum of a cone by regarding the surface as the limit of the sum of a number of trapezia.

# TRIGONOMETRY

## CHAPTER IX

## RATIOS OF ACUTE ANGLES

**The six trigonometric ratios**

If the angles of a triangle are known, then the shape of the triangle is determined, though not its size. We can draw as many triangles as we please having their angles equal to the given angles ; they will all be " similar triangles " (p. 133).

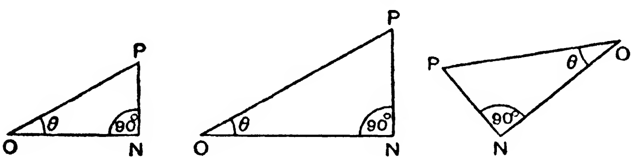



FIG. 136.

In Fig. 136 each of the triangles has $O\widehat{N}P = 90°$ and $N\widehat{O}P = \theta$. They are similar triangles and therefore their corresponding sides are proportional (p. 134) ; that is, for example, the ratio $NP/OP$ is the same for all the triangles shown, though of course if $\theta$ had a different value this ratio would be different. The ratio $NP/OP$, which depends only on the value of $\theta$, is defined as the *sine* of the angle $\theta$, written as $\sin \theta$.

There are, altogether, six ratios between the three sides of the triangle $ONP$, viz. :

$$\frac{NP}{OP}, \frac{ON}{OP}, \frac{NP}{ON}, \frac{ON}{NP}, \frac{OP}{ON}, \frac{OP}{NP}.$$

Each of these ratios is independent of the size of the triangle and depends only on the value of the angle $\theta$. They are spoken of as "the six trigonometric ratios of the angle $\theta$," and are called the *sine*, *cosine*, *tangent*, *cotangent*, *secant* and *cosecant*, respectively, of the angle $\theta$; they are written as $\sin\theta$, $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$, respectively.

The first three ratios have already been met with in Part I.

### Definitions of the ratios

We can set out the definitions of the six ratios concisely in the following form.

Draw a right-angled triangle having one of its acute angles equal to the given angle $\theta$. The side opposite to the right angle is called the hypotenuse. Now label the side which is opposite to the angle $\theta$ as "opposite side," and the remaining side as "adjacent side" (it is adjacent to the angle $\theta$). Then, using the words "opposite" and "adjacent" as short for "opposite side" and "adjacent side,"

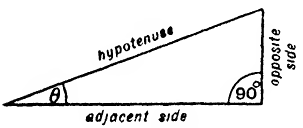



FIG. 137.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \operatorname{cosec}\theta = \frac{\text{hypotenuse}}{\text{opposite}}.$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}.$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}, \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}.$$

### Relations between the six ratios

It will be noticed from the definitions that three of these ratios are the reciprocals of the other three, viz.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

[It will serve as a help in remembering these if we observe that in each of the three pairs of reciprocals there is one ratio with the prefix "co"; thus the reciprocal of sin is *cosec*, of sec is *cos*, of tan is *cot*.]

Also, referring to Fig. 136,

$$\tan\theta = \frac{NP}{ON} = \frac{\dfrac{NP}{OP}}{\dfrac{ON}{OP}} = \frac{\sin\theta}{\cos\theta}, \text{ and hence } \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Thus all the ratios can be expressed in terms of sin $\theta$ and cos $\theta$.

We shall see later that, in the case of acute angles, any one of the ratios can be expressed in terms of any other, so that the values of all the ratios can be found if the value of *one* of them is known.

## Tables of the ratios

The values of the ratios of angles from 0° to 90° are tabulated in books of trigonometric tables under the headings "natural sines," "natural secants," etc. In some books of tables only the values of sines, cosines and tangents are given; in that case the values of the other three ratios can be found from the relations above.

Where tables of cotangents, secants and cosecants are provided, however, the student is advised to make use of them; for not only does it save time and labour to read off, for example, sec 28° directly from the table of secants, instead of looking up cos 28° and then working out $\frac{1}{\cos 28^\circ}$, but there is less risk of mistake or inaccuracy since the numerical work in compiling the tables has been done by professional computers.

*Example.*—The top of a vertical pole is 9 ft. above the ground. Find, to the nearest inch, the length of the shadow cast on the ground when the elevation of the sun is 76°.

In Fig. 138, $AB$ represents the pole, and $BC$ the shadow on the ground. We are given that

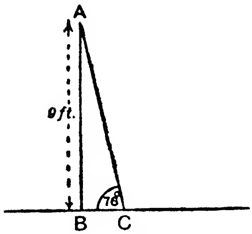



FIG. 138.

$AB = 9$ ft. and $A\widehat{C}B = 76°$.

In the right-angled triangle $ABC$,

$$\frac{BC}{AB} = \cot 76°$$

$$\therefore\ BC = AB \cot 76° = 9 \cot 76°$$

$$= 9 \times 0{\cdot}2493 \simeq 2{\cdot}244 \text{ ft.}$$

$= 2$ ft. 3 in. (to the nearest inch).

Hence the length of the shadow is 2 ft. 3 in. (to the nearest inch).

An alternative method is as follows:

$$\frac{AB}{BC} = \tan 76°$$

$$\therefore\ BC = \frac{AB}{\tan 76°} = \frac{9}{4{\cdot}0108} \simeq 2{\cdot}244 \text{ ft.} \simeq 2 \text{ ft. } 3 \text{ in.}$$

The former method is the easier, since it involves multiplying 9 by 0·2493 instead of dividing 9 by 4·0108.

As a general rule, it is best to use a ratio in which the unknown quantity ($BC$ in the above example) occurs in the numerator.

*Example.*—A straight road up a hillside makes an angle of 10° with the horizontal. How far must a man walk along the road in order to rise a vertical height of 400 ft.?

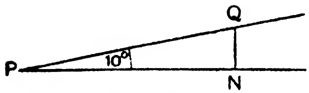



FIG. 139.

In Fig. 139, $PQ$ represents the distance the man walks along the road. His vertical rise is $NQ$. $\therefore\ NQ = 400$ ft.

Here the unknown length, which we want to find, is $PQ$, and so we try to find a ratio in which $PQ$ is the numerator and

in which the denominator is known. The only known length is $NQ$; so we take:

$$\frac{PQ}{NQ} = \text{cosec } 10^\circ$$

$$\therefore PQ = NQ \text{ cosec } 10^\circ = 400 \text{ cosec } 10^\circ$$
$$= 400 \times 5{\cdot}7588 \simeq 2304 \text{ ft.}$$

Hence the man must walk a distance of 2304 ft. along the road.

If tables of cosecants are not available, we must work from a table of sines, thus:

$$PQ = 400 \text{ cosec } 10^\circ = \frac{400}{\sin 10^\circ} = \frac{400}{0{\cdot}1736} \simeq 2304 \text{ ft.}$$

**Radian measure for angles**

Angles are ordinarily measured in practical work (e.g. on protractors, galvanometer scales, etc.) in degrees and minutes, but there is another unit for angle which is extremely useful in theoretical work. It is the angle subtended at the centre of a circle by an arc equal in length to the radius, and it is called a *radian*. It is the angle $A\hat{O}B$ in Fig. 140.

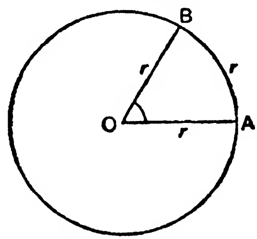



FIG. 140.

We can easily convert from degrees to radians or *vice versa*, just as we can convert from inches to centimetres or from grammes to pounds. For in a circle of radius $r$, half the circumference has a length $\pi r$ and this arc subtends an angle 180° at the centre. Hence an arc of length $r$ subtends an angle $\frac{180^\circ}{\pi}$ at the centre. Thus, by our definition of a radian,

$$1 \text{ radian} = \frac{180^\circ}{\pi} \simeq 57{\cdot}3^\circ \simeq 57^\circ\ 18'$$

(more accurately 1 radian $\simeq 57{\cdot}296^\circ \simeq 57^\circ\ 17'\ 45''$).

Hence also,

$$1^\circ = \frac{\pi}{180} \text{ radian} \simeq 0{\cdot}01745 \text{ radian.}$$

The rule for conversion is most easily remembered in the form :

$$\pi \text{ radians} = 180^\circ.$$

**Length of arc of circle**

If an arc of a circle subtends an angle $\theta$ radians (usually written as $\theta^c$) at the centre, its length is $\theta$ times that of an arc which subtends 1 radian at the centre.

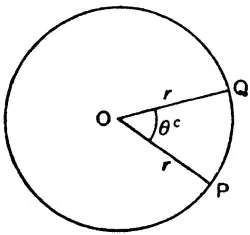



FIG. 141

Hence, if the radius of the circle is $r$,

*length of arc* $= r\theta$.

If the angle is measured in *degrees*, say $x^\circ$, then since $x^\circ = \frac{\pi x}{180}$ radians, the length of the arc is $r \cdot \frac{\pi x}{180}$.

**Area of sector of circle**

Referring to Fig. 141,

$$\frac{\text{area of sector } POQ}{\text{area of circle}} = \frac{\text{angle subtended by } PQ}{\text{angle subtended by whole circumference}}$$

$$= \frac{\theta}{2\pi}$$

$$\therefore \text{ area of sector } POQ = \pi r^2 \times \frac{\theta}{2\pi} = \tfrac{1}{2}r^2\theta.$$

Hence

$$\textit{area of sector of circle} = \tfrac{1}{2}r^2\theta.$$

*Example.*—**An endless belt passes over two pulleys, whose radii are $r_1$ and $r_2$ and whose centres are distant $d$ apart. Find the length of the belt, assuming it is taut throughout.**

The belt may be open (Fig. 142 (*a*)) or crossed (Fig. 142 (*b*)). In the former case the two pulleys rotate in the same direction, in the latter case they rotate in opposite directions.

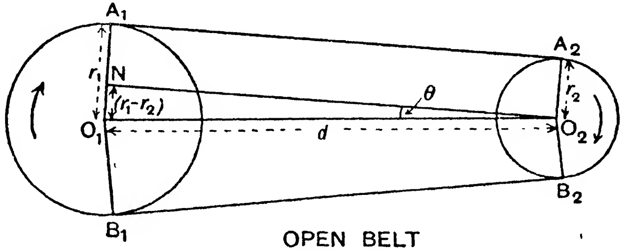



FIG. 142 (*a*).

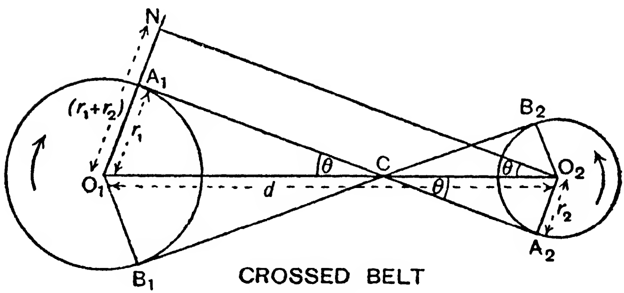



FIG. 142 (*b*).

$O_1A_1$, $O_1B_1$, $O_2A_2$ and $O_2B_2$ are the radii to the points of contact of the belt with the pulleys; $O_1A_1$ is parallel to $O_2A_2$, since each is perpendicular to the tangent $A_1A_2$. Also $O_1B_1$ is parallel to $O_2B_2$.

$O_2N$ is drawn perpendicular to $O_1A_1$, or $O_1A_1$ produced.

For an open belt $O_1N = r_1 - r_2$; for a crossed belt $O_1N = r_1 + r_2$.

Let the angle between each of the straight parts of the belt and the line of centres be $\theta$ radians.

Let the total length of the belt be $l$.

I. For open belt

$O_2\widehat{O_1}N = \left(\frac{\pi}{2} - \theta\right)$ radians.

$\therefore$ Angle of lap on larger pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians.

$\therefore$ Length of belt in contact with larger pulley $= r_1(\pi + 2\theta)$.

Also $O_1\widehat{O_2}A_2 = \left(\frac{\pi}{2} + \theta\right)$ radians.

$\therefore$ Angle of lap on smaller pulley $= 2\pi - 2\left(\frac{\pi}{2} + \theta\right) = (\pi - 2\theta)$ radians.

$\therefore$ Length of belt in contact with smaller pulley $= r_2(\pi - 2\theta)$.

Also $A_1A_2 = NO_2 = d \cos \theta$ and $B_1B_2 = A_1A_2 = d \cos \theta$.

$\therefore$ Total length of belt $= r_1(\pi + 2\theta) + r_2(\pi - 2\theta) + 2d \cos \theta$,

$$\text{i.e. } l = \pi(r_1 + r_2) + 2\theta(r_1 - r_2) + 2d \cos \theta.$$

The angle $\theta$ can be found from the relation $O_1N = O_1O_2 \sin \theta$,

$$\text{i.e. } r_1 - r_2 = d \sin \theta.$$

We can also write the expression for $l$ in the form

$$l = \pi(r_1 + r_2) + 2\theta(r_1 - r_2) + 2(r_1 - r_2) \cot \theta,$$

on substituting for $d$ in terms of $r_1$ and $r_2$.

$$\therefore\ l = \pi(r_1 + r_2) + 2(r_1 - r_2)(\theta + \cot \theta).$$

II. For crossed belt

Angle of lap on larger pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians, as before.

$\therefore$ Length of belt in contact with larger pulley $= r_1(\pi + 2\theta)$.

Also $C\widehat{O_2}A_2 = \left(\frac{\pi}{2} - \theta\right)$ radians.

$\therefore$ Angle of lap on smaller pulley $= 2\pi - 2\left(\frac{\pi}{2} - \theta\right) = (\pi + 2\theta)$ radians.

$\therefore$ Length of belt in contact with smaller pulley $= r_2(\pi + 2\theta)$.
Also $A_1A_2 = NO_2 = d \cos \theta$ and $B_1B_2 = A_1A_2 = d \cos \theta$.

$$\therefore\ l = r_1(\pi + 2\theta) + r_2(\pi + 2\theta) + 2d \cos \theta$$
$$= \pi(r_1 + r_2) + 2\theta(r_1 + r_2) + 2d \cos \theta.$$

The angle $\theta$ is now given by the relation $r_1 + r_2 = d \sin \theta$.
The alternative form for $l$ is :

$$l = \pi(r_1 + r_2) + 2(r_1 + r_2)(\theta + \cot \theta).$$

It will be noticed that, if the distance $d$ is fixed, the length of a *crossed* belt depends only on $(r_1 + r_2)$, the *sum* of the radii of the pulleys, and not on $r_1$ and $r_2$ separately. This fact is made use of in fitting step or cone pulleys for varying the velocity ratio of two shafts.

## Variation of the ratios as the angle increases from 0° to 90°

Since the hypotenuse is the longest side of a right-angled triangle, $\sin \theta$ and $\cos \theta$ are each less than 1, and their reciprocals, $\operatorname{cosec} \theta$ and $\sec \theta$, are therefore each greater than 1 for all values of $\theta$ between 0° and 90°.

If $OA$ and $OB$ are two fixed radii of a circle, at right angles to each other (Fig. 143), and if $OP$ is a radius making an angle $\theta$ with $OA$, then, as $\theta$ increases from 0° to 90°, $OP$ turns from the position $OA$ to the position $OB$.

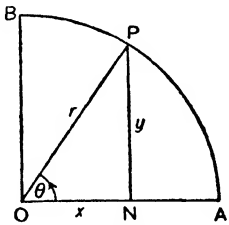


FIG. 143.

Suppose $PN$ is perpendicular to $OA$, and that the radius of the circle is $r$ units. Then if $ON = x$ units and $NP = y$ units,

$$\sin \theta = \frac{y}{r},\ \cos \theta = \frac{x}{r},\ \tan \theta = \frac{y}{x},\ \cot \theta = \frac{x}{y},\ \sec \theta = \frac{r}{x},\ \operatorname{cosec} \theta = \frac{r}{y}.$$

As $\theta$ increases from 0° to 90°, $y$ increases from 0 to $r$ and $x$ decreases from $r$ to 0. Hence $\sin \theta$ increases from 0 to 1, and $\cos \theta$ decreases from 1 to 0.

In $\tan \theta$, which is equal to $y/x$, the numerator increases and the denominator decreases, so that the fraction increases.

When $\theta=0$, $\tan\theta=0$; when $\theta$ is just less than $90°$, $y$ is nearly equal to $r$ and $x$ is nearly zero, so that $y/x$ is then very large, and as $\theta$ approaches more and more closely to $90°$, $x$ becomes smaller and smaller and $y/x$, i.e. $\tan\theta$, increases without limit. We express this briefly by saying that $\tan\theta$ tends to infinity as $\theta$ tends to $90°$, or, still more briefly, that $\tan 90°=\infty$ (read as $\tan 90°$ = infinity).

If the student looks at his tables he will find that $\tan 88°=28{\cdot}64$, $\tan 89°=57{\cdot}29$, $\tan 89°\ 30'=114{\cdot}6$, $\tan 89°\ 48'=286{\cdot}5$, $\tan 89°\ 54'=573{\cdot}0$. Also, from seven-figure tables he will find that $\tan 89°\ 58'=1718{\cdot}9$, $\tan 89°\ 59'=3437{\cdot}7$.

The other three ratios are the reciprocals of the three already considered. Now when a quantity increases its reciprocal decreases, and *vice versa*, and hence we see that, as $\theta$ increases from $0°$ to $90°$, $\cot\theta$ decreases from $\infty$ to 0, $\sec\theta$ increases from 1 to $\infty$, $\operatorname{cosec}\theta$ decreases from $\infty$ to 1.

The student can also verify these statements by considering how the ratios $\frac{x}{y}$, $\frac{r}{x}$ and $\frac{r}{y}$ vary as $P$ goes from $A$ to $B$.

**Ratios of some important angles**

The angles $30°$, $45°$ and $60°$ are important because they occur so frequently in practical applications of trigonometry, and also because their ratios can be easily found without the use of tables.

If we draw a square $ABCD$ of side 1 unit and draw one diagonal $AC$ (Fig. 144 (*a*)), then $C\widehat{A}B=A\widehat{C}B=45°$, and, from Pythagoras' theorem, $AC=\sqrt{2}$ units. From the right-angled triangle $ABC$ we can write down all the trigonometric ratios of $45°$.

For example,

$$\sin 45°=\frac{BC}{AC}=\frac{1}{\sqrt{2}},\ \tan 45°=\frac{BC}{AB}=1,\ \sec 45°=\frac{AC}{AB}=\sqrt{2}.$$

If we draw an equilateral triangle $PQR$ (Fig. 144 (b)) and draw $PM$ perpendicular to $QR$, then $PM$ bisects $QR$ and also bisects the angle $QPR$. Thus if we call the length of each side

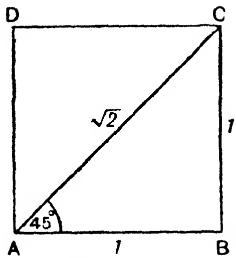



FIG. 144 (a).

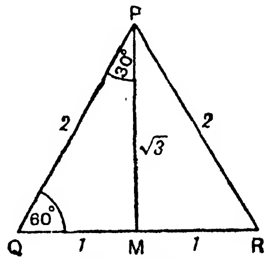



FIG. 144 (b).

of the triangle $PQR$ 2 units, $QM=1$ unit, and, from Pythagoras' theorem, $PM=\sqrt{(PQ^2-QM^2)}=\sqrt{3}$ units. The angles of the triangle $PQM$ are 60°, 90° and 30°, and from that triangle we can now write down all the trigonometric ratios of 60° and 30°.

For example, $\cos 60° = \frac{QM}{PQ} = \frac{1}{2}$, $\operatorname{cosec} 60° = \frac{PQ}{MP} = \frac{2}{\sqrt{3}}$,

$$\cot 30° = \frac{MP}{QM} = \sqrt{3},\ \sin 30° = \frac{QM}{PQ} = \tfrac{1}{2}.$$

The student should make a mental picture of the diagrams above, and should remember the values of $\sqrt{2}$ and $\sqrt{3}$, viz. $\sqrt{2} \simeq 1{\cdot}414$, $\sqrt{3} \simeq 1{\cdot}732$, so that he can convert the ratios to decimals if required.

## Note on the use of tables

Since $\cos\theta$, $\cot\theta$ and $\operatorname{cosec}\theta$ all decrease as $\theta$ increases, the numbers in the difference columns in the tables of those ratios must be subtracted, not added.

Sin $\theta$, tan $\theta$ and sec $\theta$ all increase with $\theta$, and so the numbers in their difference columns must be added.

*Example.*—Find cot 42° 8′.

From tables, cot $42° \ 6' = 1 \cdot 1067$

difference for $2' = 0 \cdot 0013$

$\therefore \cot 42° \ 8' = 1 \cdot 1067 - 0 \cdot 0013 = 1 \cdot 1054.$

*Example.*—Find sec 66° 27′.

From tables, sec $66° \ 24' = 2 \cdot 4978,$

difference for $3' = 0 \cdot 0050,$

$\therefore \sec 66° \ 27' = 2 \cdot 4978 + 0 \cdot 0050 = 2 \cdot 5028.$

We are not likely to forget whether to add or subtract if we remember that 42° 8′ being greater than 42° 6′, its cotangent must be less than that of 42° 6′.

There is an easy rule for remembering whether to add or subtract differences. If we call the ratios which have the prefix *co-* (viz. cosine, cotangent, cosecant) the "co-ratios," the rule is: *subtract differences in the case of the co-ratios*, add in the case of the other ratios.

The same rule applies also to the tables of the logarithms of the trigonometric ratios (i.e. tables of log, sines, etc.).

The student will notice that in some of the tables there are certain ranges of values of the angle for which mean differences are not tabulated. For example, in four-figure tables of natural tangents no mean differences are tabulated when the angle is greater than about 77°. The reason is that the increase in the value of tan $\theta$ when $\theta$ increases from, say, 86° to 86° 6′ is 0·37 while the increase in tan $\theta$ when $\theta$ increases from 86° 48′ to 86° 54′ is 0·57; so that an increase of 1 minute in the angle between 86° and 86° 6′ corresponds to an increase of about 0·06 in the value of tan $\theta$, which the same increase in the angle between 86° 48′ and 86° 54′ corresponds to an increase of about 0·10 in the value of tan $\theta$. Thus no number in the difference column would be correct for the whole range of values in the line marked 86°.

In such cases the student can obtain a reasonably good value for the function by using proportional parts as shown in the following example.

*Example.*—Find tan 79° 34′.

From tables, tan 79° 30′ = 5·3955

tan 79° 36′ = 5·4486

$\therefore$ Difference for 6′ = 0·0531

$\therefore$ Difference for 4′ $\simeq \frac{4}{6} \times 0{\cdot}0531 = 0{\cdot}0354$

$\therefore$ tan 79° 34′ $\simeq$ 5·3955 + 0·0354 = 5·4309.

## Exercise XXIV

**1.** Find, from tables,

(i) sin 55° 16′, (ii) tan 67° 19′, (iii) cot 48° 39′,
(iv) sec 12° 52′, (v) cos 81° 4′, (vi) cosec 33° 21′,
(vii) tan (0·571 radians), (viii) cosec (1·364 radians).

**2.** In Fig. 145 express as ratios, in terms of $a$, $b$, $c$,

(i) sin $\alpha$,
(ii) cot $\beta$,
(iii) sec $\alpha$,
(iv) tan $\alpha$,
(v) cos $\beta$,
(vi) cosec $\alpha$.

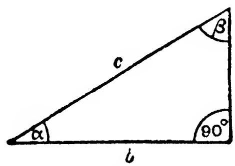
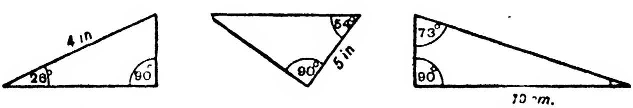



FIG. 145.

**3.** Find, without the use of tables,

sin 60°, cosec 30°, tan 60°, cosec 45°, sec 30°, cot 60°.

**4.** Find the lengths of the remaining sides of the triangles shown:




(i) (ii) (iii)

FIG. 146.

**5.** A man rows across a river in a direction making an angle of 55° with either bank. If the river is 30 yd. wide, how far does he have to row?

6. A pendulum 15 in. long swings, in a vertical plane, through an angle of 22° on either side of the vertical (Fig. 147). Find the vertical height through which its end rises.

7. In the preceding question, find the total horizontal distance through which the end moves in one complete oscillation.

8. A ladder, which leans against a vertical wall, makes an angle of 65° with the ground and reaches a window 12 ft. above the ground. Find the length of the ladder.

9. Find the area of the triangle in Fig. 148. Find also the distance of $A$ from $BC$.

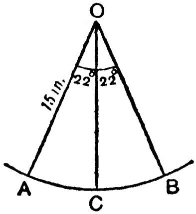



FIG. 147.

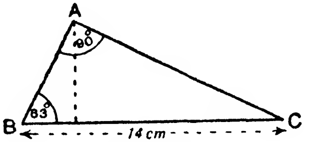



FIG. 148.

10. Find the angle $\theta$ (to the nearest minute) if:

(i) $\sin \theta = 0{\cdot}55$, (ii) $\cot \theta = 0{\cdot}164$, (iii) $\tan \theta = 2{\cdot}26$.

11. Find the angle $\alpha$ (to the nearest minute) if:

(i) $\cos \alpha = 0{\cdot}973$, (ii) $\sec \alpha = 3{\cdot}453$, (iii) $\operatorname{cosec} \alpha = 1{\cdot}75$.

12. Draw an angle whose secant is 1·8. Measure the angle, and compare your answer with that given in the tables.

13. Find the remaining angles in the triangles shown:

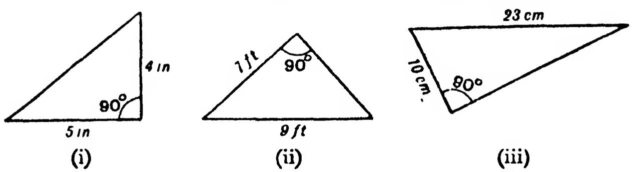



FIG. 149.

14. Verify, from tables, that $\log \sec 58° \, 24' = -\log \cos 58° \, 24'$. Explain why the relation $\log \sec \theta = -\log \cos \theta$ is true for all values of $\theta$.

**15.** A wireless mast 20 ft. high casts a shadow of length 14 ft. 6 in. on the ground. Find the angle of elevation of the sun.

**16.** A garage with a span roof has a rectangular floor of dimensions shown in Fig. 150, and the pitch of the roof is 28°. Find the area of the roof.

**17.** Water flows over a V-notch (Fig. 151) at a velocity of 40 ft. per sec. If the angle of the notch is 50° and the depth of the water is kept constant at 3 ft., find the volume which flows over in one minute.

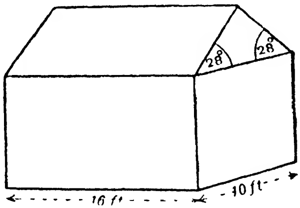


Fig. 150.

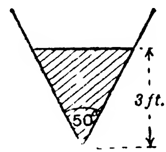


Fig. 151.

**18.** A telegraph pole of diameter 1 ft. 8 in. is strengthened by a wire cable, which passes once round the pole and is secured to the ground, as in Fig. 152. Find the length of cable required. Find also the distance of the foot of the cable from the centre of the pole.

**19.** Each leg of a step-ladder is 8 ft. long and it stands on level ground with its feet 5 ft. 6 in. apart. Find the angle which each leg makes with the ground, and the height of the top above the ground.

**20.** Find the lengths of the sides $AB$ and $BC$ of the metal plate in Fig. 153, and find the area of the plate.

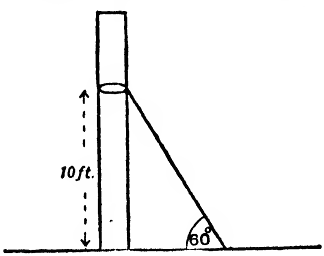


Fig. 152.

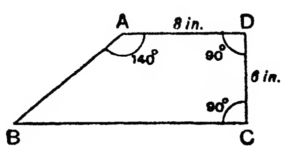


Fig. 153.

**21.** If a shell is fired with velocity $v$ ft. per sec. from a gun at an elevation $\alpha$, it reaches a height $\frac{1}{64}v^2 \sin^2 \alpha$ ft. and strikes the ground at a distance $\frac{1}{16}v^2 \sin \alpha \cos \alpha$ ft. from the gun, air resistance being neglected. If $v=1200$ find (i) the distance at which it strikes the ground when $\alpha=52°$, (ii) the angle of elevation necessary to reach a height of 9000 ft.

**22.** The two tangents from a point $P$ to a circle of radius 5 cm. are inclined at an angle of 34° to each other. Find the length of either tangent, and the distance of $P$ from the centre of the circle.

**23.** The power, $p$ watts, in an alternating-current circuit is given by $p=vi$, where $v$ is the voltage (in volts) and $i$ is the current (in amperes). If $v=200 \sin 50t$ and $i=10 \sin (50t+\frac{\pi}{4})$, the angles being in radians, find the power when $t=0{\cdot}01$.

**24.** A baulk of timber of the dimensions shown (Fig. 154) rests over a cylindrical drum of diameter 14 in. Find the heights of $A$ and $B$ above the ground.

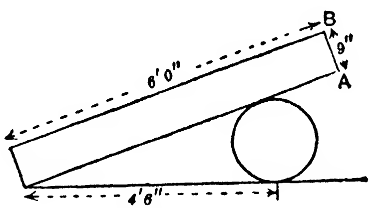



FIG. 154.

**25.** Draw the graph of $y=\sin x$ for values of $x$ from 0° to 90°.

**26.** Draw the graph of $y=\cos x$ from $x=0$ to $x=\frac{1}{2}\pi$ radians.

**27.** Draw the graph of $y=\tan x$ from $x=0°$ to $x=90°$.

**28.** A belt passes over two pulleys whose diameters are 2 ft. 2 in. and 1 ft., and whose centres are 4 ft. apart. Find the length of belt necessary (i) open, (ii) crossed.

## Complementary angles

If the sum of two angles is a right-angle the angles are said to be "complementary"; each is called the "complement"

of the other. The complement of an angle $\theta$ is therefore the angle $90° - \theta$, or $\frac{\pi^c}{2} - \theta$. (Note. $\frac{\pi^c}{2}$ means $\frac{\pi}{2}$ radians.)

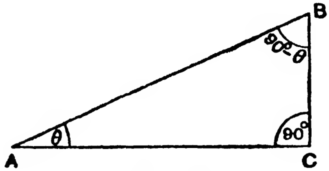



FIG. 155.

If we draw a right-angled triangle $ABC$ with $B\widehat{A}C = \theta$, then, since the three angles add up to 180°, $A\widehat{B}C = 90° - \theta$. From the figure,

$$\sin (90° - \theta) = \frac{AC}{AB} = \cos \theta ; \qquad \cos (90° - \theta) = \frac{BC}{AB} = \sin \theta ;$$

$$\tan (90° - \theta) = \frac{AC}{BC} = \cot \theta ; \qquad \cot (90° - \theta) = \frac{BC}{AC} = \tan \theta ;$$

$$\sec (90° - \theta) = \frac{AB}{BC} = \operatorname{cosec} \theta ; \quad \operatorname{cosec} (90° - \theta) = \frac{AB}{AC} = \sec \theta .$$

These are best remembered in words, thus:

*cosine of any angle = sine of its complement ;*
*cotangent of any angle = tangent of its complement ;*
*cosecant of any angle = secant of its complement.*

We can also interchange the words sine and cosine, tangent and cotangent, secant and cosecant in these relations, as is obvious since the complement of $90° - \theta$ is $\theta$ itself.

These relations show why we use the terms co-sine, co-tangent and co-secant, the prefix *co-* being the first part of the word complement.

By the aid of the three relations above we can use tables of sines, tangents and secants to find cosines, cotangents and cosecants. For example, $\cot 25° \ 12' = \tan (90° - 25° \ 12') = \tan 64° \ 48' = 2{\cdot}1251$, from the table of tangents.

### Identities

The meaning of the word *identity* in mathematics has been explained on p. 13.

The identity,

$$\sin^2\theta + \cos^2\theta \equiv 1$$

has been proved in Part I (p. 295).

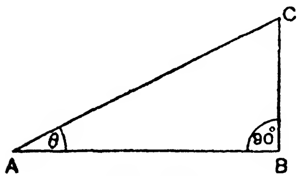



Fig. 156.

It follows at once from Pythagoras' theorem. For in Fig. 156 :

$$BC^2 + AB^2 = AC^2$$

$$\therefore \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 = 1$$

i.e. $\sin^2\theta + \cos^2\theta = 1.$

If we divide both sides of this identity by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}$$

i.e. $$\tan^2\theta + 1 = \sec^2\theta.$$

Also, dividing both sides of the first identity by $\sin^2\theta$, we have

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

i.e. $$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

As these three identities are often useful for simplifying trigonometric formulæ, we shall collect them together for reference :

$$\underline{\sin^2\theta + \cos^2\theta \equiv 1,}$$

$$\underline{1 + \tan^2\theta \equiv \sec^2\theta,}$$

$$\underline{1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta}$$

*Example.*—Prove that $\sin^2 \theta + \sin^2 \theta \tan^2 \theta \equiv \tan^2 \theta$.

L.H.S. (meaning " left-hand side " of the relation)

$$= \sin^2 \theta \, (1 + \tan^2 \theta)$$
$$= \sin^2 \theta \,.\, \sec^2 \theta$$
$$= \frac{\sin^2 \theta}{\cos^2 \theta}$$
$$= \tan^2 \theta = \text{R.H.S.}$$

*Example.*—Simplify the expression $\dfrac{1+\cos x}{\sin x} + \dfrac{\sin x}{1+\cos x}$.

$$\text{Expression} = \frac{(1+\cos x)^2 + \sin^2 x}{\sin x \, (1+\cos x)}$$
$$= \frac{1 + 2\cos x + \cos^2 x + \sin^2 x}{\sin x \, (1+\cos x)}$$
$$= \frac{2 + 2\cos x}{\sin x(1+\cos x)}, \text{ since } \cos^2 x + \sin^2 x = 1,$$
$$= \frac{2(1+\cos x)}{\sin x(1+\cos x)}$$
$$= \frac{2}{\sin x}$$
$$= 2 \operatorname{cosec} x.$$

**Given one ratio of an acute angle, to find the other ratios**

*Example.*—If $\sin \alpha = 3/5$, find $\cot \alpha$.

Since $\sin^2 \alpha + \cos^2 \alpha = 1$

$$\therefore \cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \tfrac{9}{25} = \tfrac{16}{25}$$
$$\therefore \cos \alpha = \tfrac{4}{5}$$
$$\therefore \cot \alpha = \frac{\cos \alpha}{\sin \alpha} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{\cancel{5}} \times \frac{\cancel{5}}{3} = \frac{4}{3}.$$

**The point to notice here is that we have not found the angle first, and have not used trigonometric tables at all.**

[NOTE.—We took the positive square root for cos $\alpha$ above, viz. $\frac{4}{5}$, instead of $\pm\frac{4}{5}$, because we are assuming in this chapter that all the angles with which we deal are acute.]

A more direct method is as follows :

Draw a right-angled triangle $ABC$ (Fig. 157) in which $BC = 3$ units, $AB = 5$ units. Then $\sin B\widehat{A}C = \frac{3}{5}$ and therefore $B\widehat{A}C = \alpha$.

$$\therefore \cot \alpha = \frac{AC}{BC}.$$

But, from Pythagoras' theorem,

$$AC = \sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4 \text{ units};$$

$$\therefore \cot \alpha = \tfrac{4}{3}.$$

It will be seen that it is not necessary to draw the figure accurately ; we need only draw a right-angled triangle roughly,

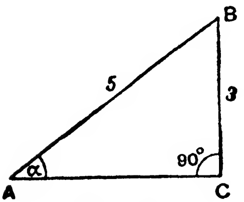



FIG. 157.

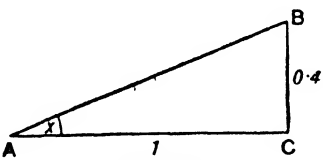



FIG. 158.

mark one of the angles $\alpha$ and call the lengths of the two sides, whose ratio is sin $\alpha$, 3 and 5, or any two convenient numbers whose ratio is 3/5. We can then find the third side from Pythagoras' theorem, and so write down any of the ratios of the angle $\alpha$.

*Example.*—If $\tan x = 0{\cdot}4$, find $\sin x$.

In $\triangle ABC$ (Fig. 158), let $B\widehat{A}C = x$.

$$\text{Then } \frac{BC}{AC} = \tan x = 0{\cdot}4.$$

Hence if $AC=1$ unit, $BC=0{\cdot}4$ units.

$$\therefore\ AB=\sqrt{1^2+(0{\cdot}4)^2}=\sqrt{1+0{\cdot}16}=\sqrt{1{\cdot}16}\simeq 1{\cdot}077.$$

$$\therefore\ \sin x=\frac{BC}{AB}\simeq\frac{0{\cdot}4}{1{\cdot}077}\simeq 0{\cdot}371.$$

## Exercise XXV

1. Verify from tables that $\cos 28°\ 36'=\sin(90°-28°\ 36')$, and that $\tan 28°\ 36'=\cot(90°-28°\ 36')$.

2. Verify the relations $\cos\theta=\sin(90°-\theta)$, etc., when $\theta=60°$, without using tables.

3. If $\sin x=\cos 25°$, find $x$.

4. Show that $\sin(90°-\alpha)\,.\,\cot(90°-\alpha)=\sin\alpha$.

5. Show that $\dfrac{\cos\left(\frac{\pi}{2}-\beta\right)}{\sec\left(\frac{\pi}{2}-\beta\right)}=\sin^2\beta$.

6. If $\sin\theta=5/13$, find $\tan\theta$.

7. If $\cos\phi=1/2$, find $\operatorname{cosec}\phi$.

8. If $\tan\alpha=3$, verify that $\dfrac{\sin\alpha-\cos\alpha}{\sec\alpha-\operatorname{cosec}\alpha}=0{\cdot}3$.

9. Find, without tables, the value of $\tan\theta+\sec\theta$ when $\sin\theta=\frac{1}{3}$.

Simplify the following expressions:

10. $\dfrac{1}{\sin^2\theta}-1$.

11. $\sin^3 A+\sin A\cos^2 A$.

12. $\dfrac{\sec\alpha-\cos\alpha}{\sin\alpha}$.

13. $\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$.

14. $\dfrac{1}{1+\sin x}+\dfrac{1}{1-\sin x}$.

15. $\dfrac{\sin^2 A}{\tan A}-\dfrac{\cos^2 A}{\cot A}$.

Prove that:

16. $\sin^4 x-\cos^4 x\equiv 1-2\cos^2 x$.

17. $\tan A+\cot A\equiv\sec A\operatorname{cosec}A$.

18. $\sec^2\theta+\operatorname{cosec}^2\theta\equiv\sec^2\theta\operatorname{cosec}^2\theta$.

19. $\sin^2 A\cos^2 B-\cos^2 A\sin^2 B\equiv\sin^2 A-\sin^2 B$.

20. $\sqrt{\dfrac{1+\sin x}{1-\sin x}} \equiv \sec x+\tan x$. [Hint.—Multiply numerator and denominator by $\sqrt{1+\sin x}$.]

21. $(\text{Sec } y-\cos y)(\text{cosec } y-\sin y) \equiv \sin y \cos y$.

**Applications of trigonometry to solid figures**

There are no new principles involved in applying trigonometry to solid figures. The student must be careful, however, to visualize the figure properly as a space figure, and must remember that right angles in space will not generally be represented by right angles in his diagram.

*Example.*—A room is 20 ft. long × 14 ft. wide × 10 ft. high, and a string is stretched from one corner of the ceiling to the opposite corner of the floor. Find the angle the string makes with the floor.

In Fig. 159, $AE$ represents the string. $AC$ is the projection of $AE$ on the floor; hence $C\hat{A}E$ is the required angle.

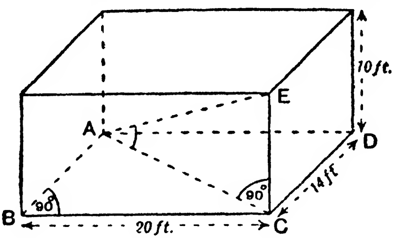



FIG. 159.

Since $A\hat{B}C$ is a right angle,

$$AC=\sqrt{AB^2+BC^2}=\sqrt{14^2+20^2}$$
$$=\sqrt{196+400}=\sqrt{596} \simeq 24{\cdot}41 \text{ ft.}$$

Also $A\hat{C}E$ is a right angle.

$$\therefore \cot C\hat{A}E=\frac{AC}{CE}=\frac{24{\cdot}41}{10}=2{\cdot}441$$

$$\therefore C\hat{A}E=\underline{22^\circ\ 17'}.$$

[The student will notice that we found $C\widehat{A}E$ here from its cotangent instead of from its tangent. We did this merely because it is easier to divide 24·41 by 10 than to divide 10 by 24·41.]

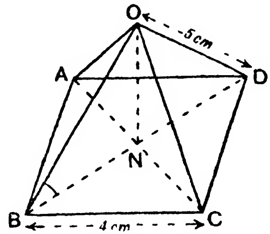


FIG. 160.

*Example.*—A right pyramid has a square base, of side 4 cm., and its slant edges are each 5 cm. long. Find the height of the pyramid and the angle between any slant edge and the base.

In Fig. 160, $ON$ is perpendicular to the base $ABCD$. From symmetry $N$ is the centre of the base.

Since $B\widehat{C}D$ is a right angle,

$$BD = \sqrt{BC^2 + CD^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2} \text{ in.}$$

$$\therefore\ BN = \tfrac{1}{2}BD = 2\sqrt{2} \text{ in.}$$

In the right-angled $\triangle OBN$ (in which $\widehat{N}$ is the right angle)

$$ON^2 = OB^2 - BN^2 = 5^2 - (2\sqrt{2})^2 = 25 - 8 = 17.$$

$$\therefore\ ON = \sqrt{17} \simeq \underline{4{\cdot}123 \text{ cm.}}$$

$BN$ is the projection of the edge $BO$ on the base. Hence angle between $BO$ and base $= O\widehat{B}N$.

$$\text{Cos } O\widehat{B}N = \frac{BN}{OB} = \frac{2\sqrt{2}}{5} = \frac{2\times 1{\cdot}414}{5} = \frac{2{\cdot}828}{5} \simeq 0{\cdot}566.$$

$$\therefore\ O\widehat{B}N \simeq \underline{55^\circ\ 32'.}$$

*Example.*—A hillside is inclined at 15° to the horizontal, and a road is to be constructed on the hillside with a gradient of 1 (vertical rise) in 12 (along the road). What must be the angle between the road and the lines of steepest slope on the hillside?

In Fig. 161, $AC$ represents a portion of the road, $AD$ and $BC$ are the lines of steepest slope through $A$ and $C$, and $ABFE$ is a horizontal plane. $DE$ and $CF$ are vertical lines.

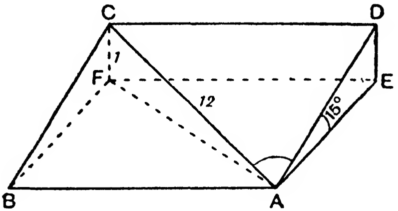



FIG. 161.

If $CF = 1$ unit, then $AC = 12$ units.

$\triangle CBF$ is a right-angled triangle and $C\widehat{B}F = 15°$.

$\therefore \dfrac{BC}{CF} = \text{cosec } 15°.$

$\therefore BC = 1 \text{ cosec } 15° = 3{\cdot}8637$ units.

The angle between the road and the lines of steepest slope $= B\widehat{C}A \; (= C\widehat{A}D)$.

$\triangle ABC$ is a right-angled triangle ($A\widehat{B}C$ being a right angle).

$\therefore \text{Cos } B\widehat{C}A = \dfrac{BC}{CA} = \dfrac{3{\cdot}8637}{12} \simeq 0{\cdot}322.$

$\therefore B\widehat{C}A \simeq \underline{71° \; 13'}.$

## Exercise XXVI

**1.** The slant height of a right circular cone is 5 in., and the diameter of the base is 8 in. Find the vertical angle and the height of the cone.

**2.** Find the angle between two diagonals of a cube.

**3.** The roof of a lean-to shed slopes at 30° and the floor is square; find the angle of slope of a diagonal of the roof.

4. Fig. 162 represents a right triangular prism whose ends are equilateral triangles of side 6 in., and $BCD$ is a section by a plane inclined at 40° to the ends. Find the lengths of $AD$ and $CD$, and the area of the section $BCD$.

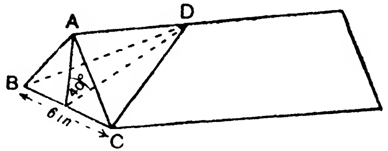



FIG. 162.

5. In Question 4, find the angle between $CD$ and each of the faces of the prism through $BC$.

6. An electric light pendant in the shape of a hemispherical bowl of radius 8 in. is suspended from a point in the ceiling by three chains each 12 in. long, attached to points at equal distances apart on the rim. Find (i) the angle between each chain and the vertical, (ii) the angle between two chains.

7. The top and base of a reservoir embankment (Fig. 163) are squares of sides 180 yd. and 200 yd. respectively, and the vertical height is 15 ft. Find the slopes of the bank and of the slant edges.

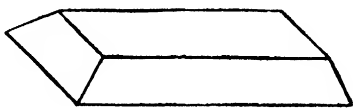


FIG. 163.

8. A hillside faces due South and has a slope of 30°. Find the gradient of a path on the hillside running in the direction N.W.

9. A vertical wall, facing East, is 16 ft. long and 5 ft. high. Find the area of its shadow on the ground when the sun is in the direction S.E. at an elevation of 60°.

## Harder problems

In some problems the unknown quantity (length or angle) cannot be found directly. We have to denote it by a symbol, such as $x$ or $\theta$, and then write down what is stated in the

problem in terms of this unknown. We may then obtain an equation which we can solve for the unknown.

*Example.*—An aeroplane is observed at the same time by two anti-aircraft batteries, distant 1 mile apart, to be at elevations of 20° and 14°. Assuming that the aeroplane is travelling directly towards the two batteries, find its height and its horizontal distance from the nearer battery.

In Fig. 164, $A$ represents the aeroplane, $B$ and $C$ the two batteries. $AD$ is vertical.

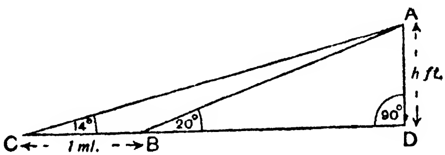



FIG. 164.

Using the foot as our unit of length, let $AD=h$. Then, from the right-angled triangle $ACD$,

$$\frac{CD}{h}=\cot 14^\circ \qquad \therefore\ CD=h\cot 14^\circ\,;$$

and, from the right-angled triangle $ABD$,

$$\frac{BD}{h}=\cot 20^\circ \qquad \therefore\ BD=h\cot 20^\circ.$$

$$\therefore\ CB=CD-BD=h(\cot 14^\circ-\cot 20^\circ).$$

But we are given that $CB=1$ ml. $=5280$ ft.

$$\therefore\ h(\cot 14^\circ-\cot 20^\circ)=5280$$

i.e. $$h(4{\cdot}0108-2{\cdot}7475)=5280$$

$$1{\cdot}2633h=5280$$

$$h=\frac{5280}{1{\cdot}2633}\simeq 4180.$$

Hence the height of the aeroplane is <u>4180 ft.</u> (approx.).

Its horizontal distance from $B$ is $BD$

| | No. | Log |
|---|---|---|
| $BD = h \cot 20°$ | | |
| $= 4180 \cot 20°$ | 4180 | 3·6212 |
| $= 11480$ ft. | cot 20° | 0·4389 |
| $= 3827$ yd. $= \underline{2 \text{ ml. } 307 \text{ yd.}}$ | 11,480 | 4·0601 |

*Example.*—A man at the top of a mountain observes, by using a theodolite, that the angles of depression of two landmarks, one due South and the other due East, are 5° 18′ and 7° 33′ respectively. He finds from a map that the landmarks are both at 100 ft. above sea level and 10·7 miles apart. What is the height of the mountain ?

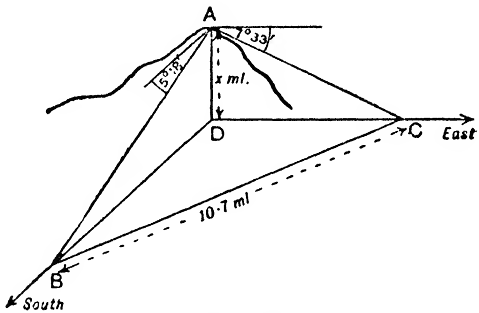



Fig. 165.

In Fig. 165, $A$ is the man ; $B$, $C$ are the two landmarks ; $BDC$ is a horizontal plane and $AD$ is vertical.

$B\widehat{A}D = 90° - 5° \, 18' = 84° \, 42'$ ; $C\widehat{A}D = 90° - 7° \, 33' = 82° \, 27'$.

Let $AD = x$ miles.

Then $\dfrac{BD}{x} = \tan 84° \, 42'$. $\quad \therefore BD = x \tan 84° \, 42'$ miles,

$\dfrac{CD}{x} = \tan 82° \, 27'$ $\quad \therefore CD = x \tan 82° \, 27'$ miles.

Since $\widehat{BDC}$ is a right angle, $BD^2 + CD^2 = BC^2$

$\therefore x^2(\tan^2 84° 42' + \tan^2 82° 27') = 10{\cdot}7^2$

$x^2(116{\cdot}2 + 56{\cdot}94) = 10{\cdot}7^2$

$173{\cdot}14\, x^2 = 10{\cdot}7^2$

$$x = \frac{10{\cdot}7}{\sqrt{173{\cdot}14}} = 0{\cdot}8132$$

$\therefore AD = 0{\cdot}8132$ ml. $= 4293$ ft.

Hence the height of the mountain $= 4293 + 100$

$= \underline{4393 \text{ ft.}}$

| *No.* | *Log* |
|---|---|
| $\tan^2 84° 42'$ | $2 \times 1{\cdot}0326 = 2{\cdot}0652$ |
| $116{\cdot}2$ | $2{\cdot}0652$ |
| $\tan^2 82° 27'$ | $2 \times 0{\cdot}8777 = 1{\cdot}7554$ |
| $56{\cdot}94$ | $1{\cdot}7554$ |
| $\frac{10{\cdot}7}{\sqrt{173{\cdot}14}}$ | $1{\cdot}0294$ |
| | $\frac{1}{2}(2{\cdot}2384) = 1{\cdot}1192$ |
| $x$ | $\bar{1}\,9102$ |

## Exercise XXVII

**1.** A straight tunnel $AB$ is bored horizontally through a mountain. The distance over the mountain is $6\frac{1}{4}$ miles, and the sides of the mountain slope at angles of 12° and 21°. Find the length of the tunnel. [Hint.—First find the height of $C$ above $AB$.]

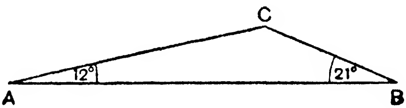



Fig. 166.

**2.** The angles of elevation of the top of a building from two windows of a house opposite are 44° 30′ and 37° 8′. If the windows are 15 ft. and 40 ft. above street level, what is the height of the building ?

**3.** A man in a boat observes that two lighthouses $P$ and $Q$ bear N. 20° W. and N. 32° E., and from his map he finds that $Q$ is 10 miles due east of $P$. Find his distance from $P$.

**4.** An aeroplane is observed at the same instant from two places $A$ and $B$, five miles apart, at elevations of 14° and 10°, being then vertically above some point between $A$ and $B$. One minute later it is vertically above $B$. Find its height (in feet) and its speed (in ml. per hr.).

5. A surveyor, who wants to find the width of a river, observes the angle of elevation of the top of a tree on the bank directly opposite to be $8^\circ\ 18'$. He then walks downstream a distance of 50 yd. and finds the elevation to be $5^\circ\ 30'$. What is the width of the river ?

6. A wall 5 ft. high runs parallel with the side of a house, and a scaffold pole 17 ft. 6 in. long resting over the wall, with its foot on the ground and the top against the house, is inclined at $34^\circ$ to the horizontal (the vertical plane of the pole being perpendicular to the wall). Find the distance between the wall and the house.

## Miscellaneous Exercise XXVIII

1. A cotter 5 in. long is 1 in. wide at one end and $1\frac{1}{4}$ in. wide at the other end (Fig. 167). Find the angle of taper.

2. The lid of a desk, hinged along the top edge, is 18 in. from front edge to hinge, and slopes at $6^\circ$ to the horizontal when closed. Through what vertical distance does the front edge rise when the lid is opened through $40^\circ$ ?

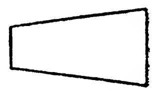


FIG. 167.

3. In a reciprocating engine (Fig. 168) the crank $OA$ is 2 ft. long and the connecting rod $AB$ is 10 ft. long. If $\widehat{AOB}=72^\circ$, find (i) the angle $\phi$, (ii) the length $OB$, (iii) the distance of the crosshead $B$ from top-dead-centre.

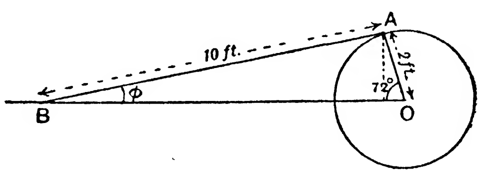



FIG. 168.

4. The pressure, $p$ lb. per sq. in. between the faces of a cone clutch is given by the formula $p=\dfrac{6F}{\pi bD}$ cosec $\alpha$, where $b$ in. is the width of the face, $\alpha$ the cone angle, $D$ in. the mean diameter and $F$ lb. the force exerted by the lever. Find, by logarithms. the value of $b$ required when $\alpha=8^\circ\ 15'$, $D=14\frac{5}{8}$, $F=600$, $p=250$.

5. Fig. 169 shows the cross-section of a railway cutting which is 60 yd. long. Find the volume of earth removed.

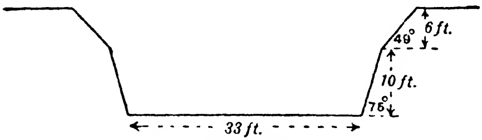



FIG. 169.

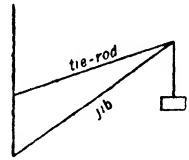



FIG. 170.

6. The jib of a crane (Fig. 170) is 14 ft. long and inclined at 54° to the vertical. The tie-rod is inclined at 72° to the vertical; find its length.

7. Vertical borings are made at points 300 yd. apart and coal is found at depths of 640 ft. and 725 ft. Find the dip of the coal seam to the nearest degree.

8. In the framework shown in Fig. 171, $AB = 16$ ft. Find the lengths of the sloping members and of $CD$.

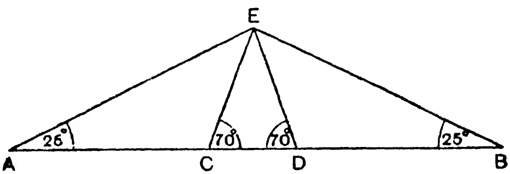



FIG. 171.

9. In the roof truss shown in Fig. 172, $AC = BC = 15$ ft., $AD = BE = 8$ ft. 6 in., $DE = 8$ ft., and the distance between the supports $A$ and $B$ is 24 ft. Find all the angles.

10. The jib of a navvy-crane (see Fig. 170) is 21 ft. long and its inclination to the horizontal can be varied between 30° and

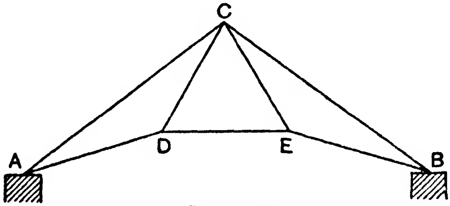



FIG. 172.

60°. The crane can turn completely round its vertical axis. Find the surface area of the ground which can be excavated. (Neglect the dimensions of the bucket.)

**11.** A surveyor's field-book reads as follows:

*Yards.*

| | | |
|---|---|---|
| | To $B$ | |
| | 435 | |
| To $C$ 85 | 385 | |
| | 185 | 150 to $E$. |
| To $D$ 140 | 120 | |
| | From $A$ | |

$B$ is due north of $A$. Find the bearings of $D$ and $E$ from $C$. [Note.—If a line $PQ$ points in the direction N. 60° E., $Q$ is said to bear N. 60° E. from $P$.]

**12.** A tunnel $AB$ is constructed through a mountain ridge (Fig. 173). $A$, $B$, $C$ are 900, 1200, 1700 ft., respectively, above sea-level, and the sides $AC$, $BC$ of the ridge slope at 27° and 34° to the horizontal. Find the gradient and the length of the tunnel.

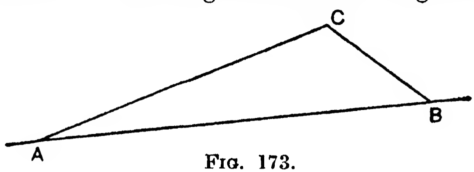



FIG. 173.

**13.** Is $\sin^2\theta = 1 - \cos\theta$ an identity or an equation? If it is an equation, solve it for $\theta$.

**14.** The E.M.F. ($e$) generated in a coil of cross-sectional area $A$ rotating with angular velocity $\omega$ about an axis perpendicular to a magnetic field of strength $H$ is given by $e = 10^{-8}\omega HA \sin \omega t$. Calculate, by logarithms, the value of $e$ when $t = 0{\cdot}006$, if $\omega = 50\pi$, $H = 30$ and $A = 140$.

**15.** A toggle joint (Fig. 174) consists of two rods $OA$, $AB$, each 15 in. long, hinged at $A$. The end $O$ is fixed and $B$ can

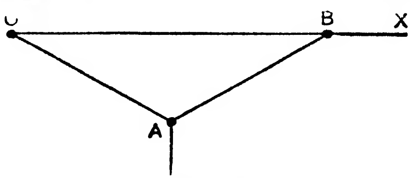



FIG. 174.

slide along the line $OX$. Initially $OB = 26$ in. Find the decrease in the angle $OAB$ when $B$ is pushed 6 in. towards $O$.

16. Two elevations of a hopper are shown in Fig. 175. Find the slope of the edge $AB$.

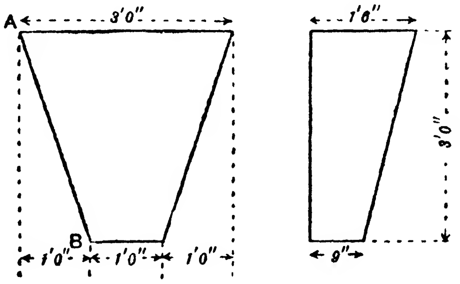



FIG. 175.

# CHAPTER X

## ANGLES OF ANY MAGNITUDE. PERIODIC FUNCTIONS

### Angles of any magnitude

If a rod turns about one end, in one revolution it turns through 360°, in two revolutions through 720°, in one and a half revolutions through 540°, and so on. We have to distinguish between the two directions of rotation, since if it rotates through 60° in one direction its final position will be different from that which it would occupy if it rotated through 60° in the opposite direction. We therefore regard rotations in the *anti-clockwise* direction as positive, and rotations in the clockwise direction as negative. Thus a rotation of 220° means turning through 220° in the anti-clockwise direction (Fig. 176); a rotation of $-60°$ means turning through 60° in the clockwise direction (Fig. 177).

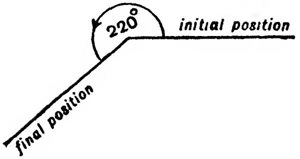



FIG. 176.

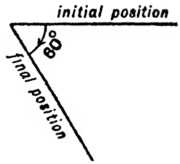



FIG. 177.

**Definitions of the trigonometric ratios of angles of any magnitude**

Let $X'OX$, $Y'OY$ be two lines at right angles, and suppose a line $OP$ rotates about $O$ from the direction $OX$ through an angle $\theta$ into the position $OP$.

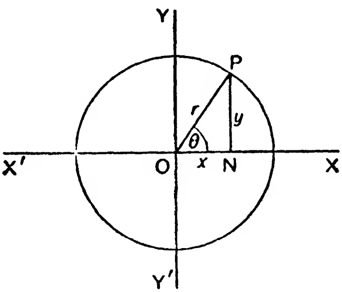



FIG. 178.

Let $x$ and $y$ be the co-ordinates of $P$ referred to the axes $OX$ and $OY$; i.e. $x$, $y$ are the algebraic values of $ON$, $NP$, respectively, $x$ being positive if $P$ lies to the right of $Y'OY$ and $y$ being positive if $P$ lies above $X'OX$.

If $OP = r$ (always regarded as positive), we define $\sin\theta$, $\cos\theta$, etc., as follows:

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},$$

and their reciprocals,

$$\operatorname{cosec}\theta = \frac{r}{y}, \qquad \sec\theta = \frac{r}{x}, \qquad \cot\theta = \frac{x}{y}.$$

We take these as our definitions of the sine, cosine, etc., of any angle whatever its magnitude, whether positive or negative. When the angle lies between 0° and 90°, they agree with the definitions previously used (in Chap. IX).

The lines $X'OX$, $Y'OY$ divide the plane of the paper into four quadrants, which we number in the order in which they would be described in a positive rotation (Fig. 179).

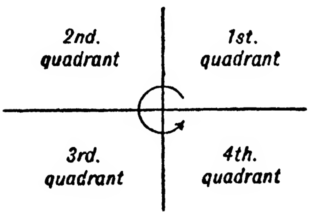



FIG. 179.

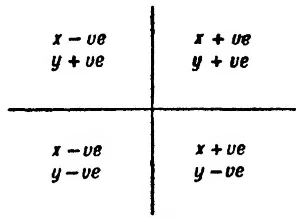



FIG. 180.

$x$ is positive in the first and fourth quadrants, negative in the second and third; $y$ is positive in the first and second quadrants, negative in the third and fourth.

Since $r$ is always counted as positive, $\sin\theta$ has the same sign as $y$, and $\cos\theta$ has the same sign as $x$. The signs of $\sin\theta$ and $\cos\theta$ in the four quadrants are therefore as shown in Fig. 181.

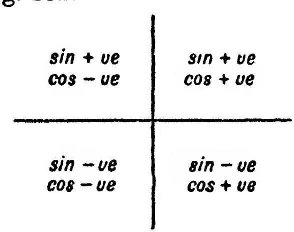



FIG. 181.

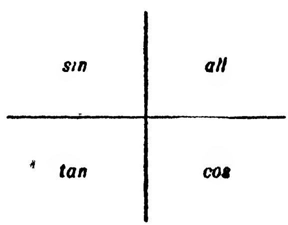



FIG. 182.

Sin $\theta$ is positive when $OP$ lies in the first or second quadrant, negative when it lies in either of the other two quadrants. Cos $\theta$ is positive in the first and fourth quadrants and negative in the other two.

Since tan $\theta = y/x$, tan $\theta$ is positive when $x$ and $y$ have the same signs (i.e. both positive or both negative) which occurs when $OP$ lies in the first or third quadrant. In the second and fourth quadrants $x$ and $y$ have opposite signs (one positive, the other negative) and therefore tan $\theta$ is negative in those two quadrants.

Since the reciprocal of a positive number is positive, and the reciprocal of a negative number is negative, the ratios cosec $\theta$, sec $\theta$ and cot $\theta$ have the same signs as their reciprocals, viz. sin $\theta$, cos $\theta$ and tan $\theta$.

The signs of the six ratios are most easily remembered by Fig. 182, where the word in any particular quadrant signifies that, in that quadrant, that ratio and its reciprocal are positive, and that all the other ratios are negative. For example, in the first quadrant all the ratios are positive ; in the fourth quadrant cos $\theta$ and its reciprocal sec $\theta$ are positive, all the others are negative.

Fig. 182 may be " read " in an anti-clockwise (i.e. positive) direction, beginning with the first quadrant, as " *all*, *sin*, *tan*, *cos*."

It will be noticed that each of the ratios is positive in two quadrants, and negative in two quadrants.

*Example.*—Find the value of cos 153°.

Here $OP$ lies in the second quadrant (Fig. 183), and $x$ is negative,

i.e. $x = -ON$,

$$\therefore \cos 153° = \frac{x}{r} = -\frac{ON}{OP} = -\cos N\widehat{O}P$$

$$= -\cos(180° - 153°) = -\cos 27°$$

$$= -0{\cdot}8910.$$

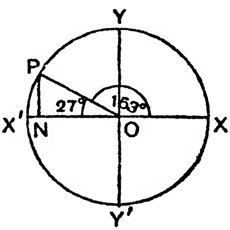



FIG. 183.

*Example.*—Find the value of tan 240°.

$OP$ lies in the third quadrant (Fig. 184); $x$ and $y$ are both negative,

$$\text{i.e. } x = -ON,\ y = -NP$$

$$\therefore \tan 240° = \frac{y}{x} = \frac{-NP}{-ON} = +\frac{NP}{ON}$$

$$= \tan N\widehat{O}P = \tan 60° = \sqrt{3} = 1{\cdot}7321.$$

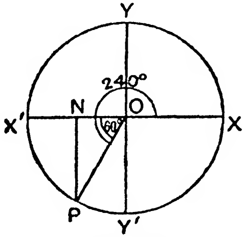



FIG. 184.

If the arm $OP$ lies in the first quadrant we usually speak of the *angle* $XOP$ as lying in the first quadrant, and so for the other quadrants.

Thus, we should say that the angle 153° lies in the second quadrant and that the angle 240° lies in the third quadrant.

### Method for finding any trigonometric ratio of any angle

We notice from the two examples above that cos 153° is *numerically* equal to cos 27°, and tan 240° is numerically equal to tan 60°, and in the same way it will be seen that the sine, cosine, etc., of *any* angle $X\widehat{O}P$ are *numerically* equal to the sine, cosine, etc., of the acute angle that $OP$ makes with the line $X'OX$. The sign to be attached to the ratio is determined by the signs of $x$ and $y$, and may be obtained from the "*all, sin, tan, cos*" rule. Thus we have the following rule for finding the sine, cosine, etc., of any angle.

*Rule.*—

(1) Estimate, by a rough sketch, in which quadrant the angle lies, and determine the sign of the ratio from the " all, sin, tan, cos " rule.

(2) Find the acute angle between $OP$ and $X'OX$ and write down the value of the corresponding ratio of that angle from tables.

After some practice the student will find that he need not draw even a rough sketch, but will be able to estimate the quadrant and the angle mentally.

*Example.*—Find sin 460°.

$460° = 360° + 100° =$ one revolution $+ 100°$.

The angle therefore lies in the second quadrant, and its sine is positive.

The acute angle required is 80° (Fig. 185).

$\therefore \sin 460° = +\sin 80° = 0{\cdot}9848.$

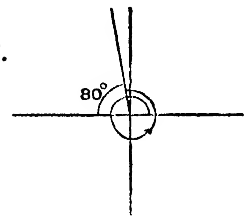



FIG. 185.

*Example.*—Find cot ( – 54°).

– 54° lies in the fourth quadrant.

Cot $\theta$ is the reciprocal of tan $\theta$ and is therefore negative in the fourth quadrant.

$\therefore \cot(-54°) = -\cot 54° = -0{\cdot}7265.$

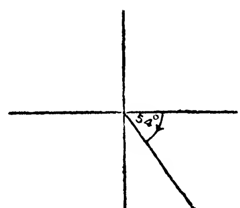



FIG. 186.

NOTE.—It will be observed that we can add, or subtract, any multiple of 360° (or $2\pi$ radians) without affecting the trigonometric ratios of the angle.

Thus in the example above, $\sin 460° = \sin (460° - 360°) = \sin 100°$.

Also, addition or subtraction of any multiple of 180°

(or $\pi$ radians) does not alter the *numerical values* of the trigonometric ratios, though it may change their sign. The sign can readily be found from the "*all*, *sin*, *tan*, *cos*" rule.

For example, $\cos 230° = -\cos 50°$; $\tan 230° = \tan 50°$.

*Example.*—The current, $i$ ampères, in a circuit after $t$ secs. is given by the formula $i = 5 \sin 100\pi t$, the angle being in radians. Find the current after 0·231 sec.

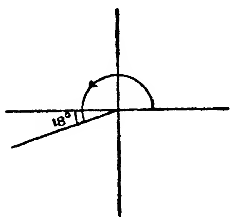



FIG. 187.

When $t = 0{\cdot}231$,

$$i = 5 \sin 23{\cdot}1\pi = 5 \sin (22\pi + 1{\cdot}1\pi)$$

$$= 5 \sin 1{\cdot}1\pi \text{ (see note above)}$$

$$= 5 \sin (1{\cdot}1 \times 180)°, \text{ since } \pi \text{ radians} = 180°,$$

$$= 5 \sin 198°.$$

198° is in the third quadrant, and its sine is negative.

$$\therefore i = 5 \times (-\sin 18°) = -5 \sin 18°$$

$$= -5 \times 0{\cdot}3090 = -1{\cdot}545 \text{ ampères.}$$

NOTE.—When angles are expressed without units, the units are understood to be radians.

## Exercise XXIX

State whether the following ratios are positive or negative:

1. $\sin 172°$, $\sin 315°$, $\cos \dfrac{5\pi}{4}$, $\tan 220°$, $\cos(-31°\ 8')$.

2. $\sin 600°$, $\cot 195°$, $\sin\left(-\dfrac{3\pi}{4}\right)$, $\sec 140°$, $\tan \dfrac{7\pi}{8}$.

Express each of the following as the ratio of an acute angle, with the proper sign (+ or −):

3. $\cos 252°$, $\sin 116°$, $\sin(-10°)$, $\tan 187°\ 30'$.

4. $\sin\frac{5\pi}{2}$, $\cos(-248°)$, $\tan 94° 15'$, $\cos\frac{13\pi}{4}$.

5. $\tan(-460°)$, $\sec 293° 42'$, $\sin 1{\cdot}82$, $\operatorname{cosec} 214°$.

Find the values of:

| | | |
|---|---|---|
| 6. Sin 140°. | 7. Sin 310° 36′. | 8. Cos 212°. |
| 9. $\cos\frac{7\pi}{4}$. | 10. Tan 136° 15′. | 11. Tan 195° 54′. |
| 12. Sin 6. | 13. $\cos\frac{5}{8}\pi$. | 14. Sec 225°. |
| 15. Cot 98° 30′. | 16. Cosec 152°. | 17. Sin (−570°). |

Verify the following relations for any acute angle $\theta$ (by drawing a rough sketch):

18. $\sin(180° - \theta) = \sin\theta$, $\cos(180° - \theta) = -\cos\theta$, $\tan(180° - \theta) = -\tan\theta$.

19. $\sin(180° + \theta) = -\sin\theta$, $\cos(180° + \theta) = -\cos\theta$, $\tan(180° + \theta) = \tan\theta$.

20. $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.

21. By substituting $(-\theta)$ for $\theta$ in the relations $\sin(90° - \theta) = \cos\theta$, etc., on p. 209, and using the results of the previous question, prove that $\sin(90° + \theta) = \cos\theta$, $\cos(90° + \theta) = -\sin\theta$, $\tan(90° + \theta) = -\cot\theta$.

22. The voltage in a certain circuit at time $t$ is $240 \sin 400t$. Find the voltage when $t = 1/80$.

23. The valve displacement, $d$ ft., in a certain machine is given by the formula

$$d = 3{\cdot}5 + 4{\cdot}2 \sin\theta + 1{\cdot}75 \cos\theta,$$

where $\theta$ is the angle through which the crank has turned. Find the displacement when the crank has turned through 130°.

24. In which quadrant does the angle $\theta$ lie if:

(i) $\sin\theta$ is positive and $\cos\theta$ is negative?
(ii) $\sin\theta$ is negative and $\cos\theta$ is positive?
(iii) $\tan\theta$ is positive and $\cos\theta$ is negative?

25. Evaluate $\cos(3t + 0{\cdot}785)$ when $t = 0{\cdot}512$.

## Graph of sin $\theta$

By constructing a table of values of $\sin\theta$ for values of $\theta$ at intervals of, say, 15°, we can draw the graph of $\sin\theta$ for any range of values of $\theta$ that we please.

There is, however, a very simple and useful method of obtaining the graph.

If in Fig. 188 the rotating line $OP$ is of length 1 unit, then $\sin\theta = \dfrac{NP}{OP} = NP$. Take a point $O'$ on $OA$ produced and let $O'X$ be taken as our axis for $\theta$; mark off on $O'X$ a scale of

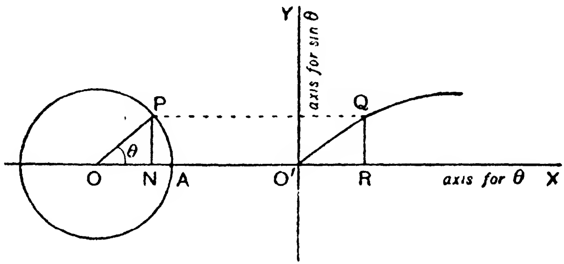



FIG. 188.

values for $\theta$. If $R$ is the point corresponding to the value $\theta = A\widehat{O}P$, the ordinate $RQ$ on our graph has to be equal to $\sin\theta$, i.e. equal to $NP$. If, therefore, we draw $PQ$ parallel to $OA$, and if this line cuts the ordinate through $R$ in $Q$, then $Q$ will be a point on the graph of $\sin\theta$.

By taking a number of different positions for $OP$ and corresponding points $R$ we get a number of points on the graph of $\sin\theta$. We then join these points by a smooth curve.

In Fig. 189 values of $\theta$ have been taken at intervals of 30° from $-30°$ to 480°. The lines $OP_0$, $OP_1$, $OP_2$, $OP_3$, . . .

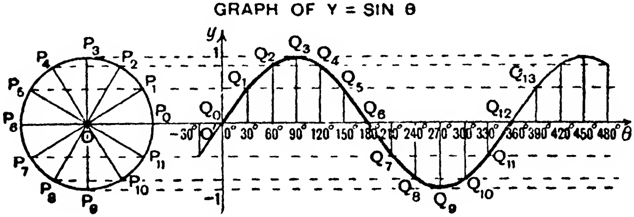



FIG. 189.

are the positions of the radius when $\theta = 0°, 30°, 60°, 90°, \ldots$, and $Q_0, Q_1, Q_2, Q_3, \ldots$ are the tops of the ordinates corresponding to those values.

By allowing the radius to rotate through an angle greater than 480° we could draw the graph for values of $\theta$ as large as we please, and by considering the radius rotating in the opposite direction we could obtain the graph for negative values of $\theta$.

It is clear from the way in which the graph is drawn that the values of $y$ are repeated during each successive revolution of the radius. For example, the value of $y$ when $\theta = 420°$ is the same as when $\theta = 60°$, and the portion of the graph between $\theta = 330°$ and $\theta = 480°$ is identical with that between $\theta = -30°$ and $\theta = 120°$. The complete graph therefore consists of a succession of waves, repeated indefinitely in both directions. The thick line in the graph indicates one complete wave.

### Graph of cos $\theta$

If $\theta$ is any acute angle, $90° + \theta$ lies in the second quadrant. Let $A\widehat{O}P = 90° + \theta$ in Fig. 190. Then $B\widehat{O}P = \theta$ and, since $NP$ is parallel to $OB$, $N\widehat{P}O = \theta$.

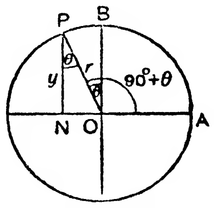



FIG. 190.

$$\sin (90° + \theta) = \frac{y}{r} = \frac{NP}{OP}, \text{ since } y \text{ is positive,}$$

$$= \cos N\widehat{P}O = \cos \theta.$$

(See also Exercise XXIX, Question 21.)

We have proved this relation only when $\theta$ is an acute angle, but it can be shown to be true for all values of the angle $\theta$. (The student could verify this for himself by considering $\theta$ to lie in each of the four quadrants in turn.)

The ordinate of the cosine graph for any value of $\theta$ is therefor equal to the ordinate of the sine graph for the value $90° + \theta$. This means that the graph of $y = \cos \theta$ can be obtained from

the graph of $y = \sin\theta$ by merely moving the origin and the $y$-axis through 90° to the right; or, what is equivalent, by keeping the origin and $y$-axis fixed and moving the sine curve bodily through 90° to the left. We thus obtain the graph in Fig. 191.

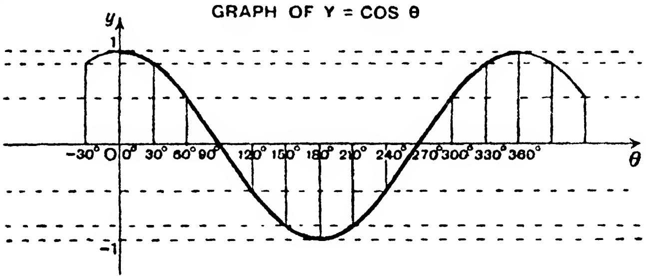



FIG. 191.

The cosine curve, being merely a displaced sine curve, also consists of a succession of waves, each of width 360° (or $2\pi$ radians). One complete wave, from $\theta = 0°$ to $\theta = 360°$, is indicated by the thick line in the graph.

## Graphs of tan θ and cot θ

To draw the graph of $\tan\theta$ we construct a table of values from a book of trigonometric tables and plot the points. So also for the graph of $\cot\theta$. The graphs are shown in Figs. 192 and 193.

It will be noticed that there are breaks in the curve $y = \tan\theta$ at $\theta = 90°$, 270°, etc., and at $\theta = -90°$, $-270°$, etc. We say that the curve is "discontinuous" at those points, in contrast to the curve of $\sin\theta$ which is a continuous curve. The curve of $\cos\theta$ is also continuous, but that of $\cot\theta$ is discontinuous at $\theta = 0°$, 180°, 360°, etc., and at $\theta = -180°$, $-360°$, etc.

The curves $y = \sin\theta$ and $y = \cos\theta$ lie entirely between the two lines $y = -1$ and $y = 1$, but the curves $y = \tan\theta$ and $y = \cot\theta$ stretch indefinitely upwards and downwards.

GRAPH OF $y = \tan\theta$

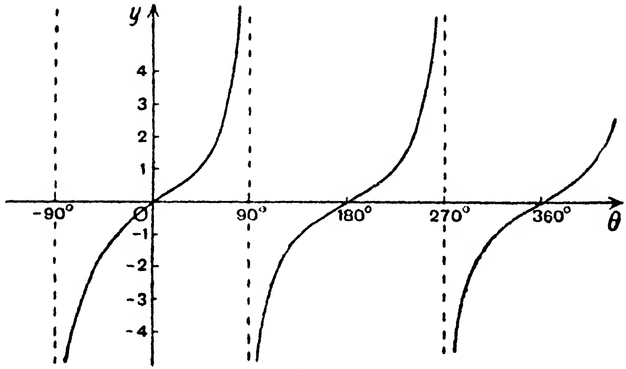



Fig. 192.

GRAPH OF $y = \cot\theta$

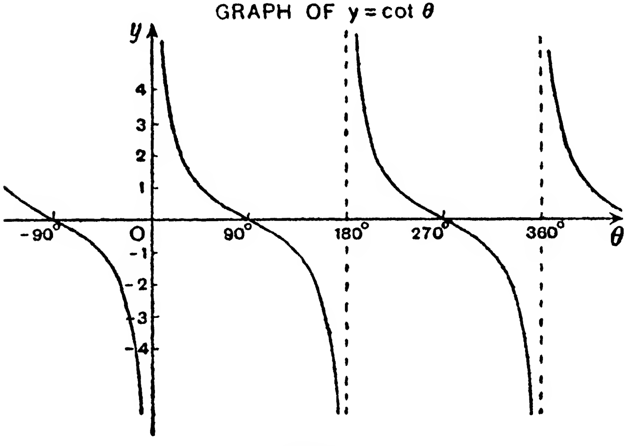



Fig. 193.

As $\theta$ approaches $90°$ from the left, i.e. by gradually increasing values, $\tan\theta$ increases indefinitely; for $\tan 88° = 28{\cdot}64$, $\tan 89° = 57{\cdot}29$, $\tan 89°\ 30' = 114{\cdot}6$, $\tan 89°\ 48' = 286{\cdot}5$,

tan 89° 54′ = 573·0, and between 89° 54′ and 90° the value of tan $\theta$ increases without limit; we say briefly that tan $\theta$ approaches $\infty$. On the other hand, as $\theta$ approaches 90° from the right, i.e. by decreasing values, we find that tan 92° = −28·64 (being negative because 92° is in the second quadrant), tan 91° = −57·29, tan 90° 30′ = −114·6, tan 90° 12′ = −286·5, tan 90° 6′ = −573·0, and between 90° 6′ and 90° the value of tan $\theta$ becomes larger and larger numerically, but is still negative. We say briefly that tan $\theta$ approaches $-\infty$. Thus when $\theta$ is just less than 90°, tan $\theta$ is a very large positive number; when $\theta$ is just greater than 90°, tan $\theta$ is a very large negative number. We cannot give any definite value to tan 90°, since we might regard it equally well as being $+\infty$ or $-\infty$.

This fact need not worry the student, and it will cause no difficulty in practice, so long as he realizes how the graph behaves near the points where the breaks occur.

### Graphs of sec $\theta$ and cosec $\theta$

These graphs do not occur so frequently in practice as the graphs of the other ratios. The graph of cosec $\theta$ is shown in Fig. 194.

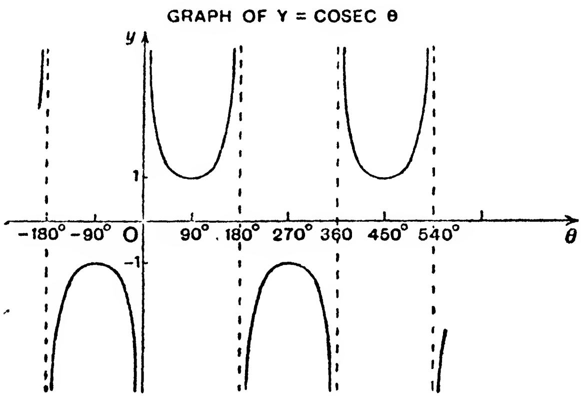



FIG. 194.

Since $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{\sin(90° + \theta)} = \operatorname{cosec}(90° + \theta)$, the graph of $\sec\theta$ is obtained by moving the curve $y = \operatorname{cosec}\theta$ bodily through 90° to the left.

Sec $\theta$ and cosec $\theta$ are both numerically greater than 1 (i.e. greater than $+1$ or less than $-1$) for all values of $\theta$.

### Periodic functions

We have seen that if we add 360°, or 720°, or any exact multiple of 360°, to an angle $\theta$, the value of $\sin\theta$ is unaltered. The values of $\sin\theta$ repeat themselves at intervals of 360°.

A function which repeats itself at regular intervals is called a *periodic function*, and the interval between two successive repetitions is called the *period* of the function. Thus $\sin\theta$ is a periodic function of $\theta$, its period being 360°, or $2\pi$ radians.

Cos $\theta$ also is a periodic function of $\theta$ with period $2\pi$ radians. Tan $\theta$ also repeats itself every 360°, but it repeats itself twice in an interval of 360°, as we can see from the graph between, say, $-90°$ and $+270°$. The *smallest* interval in which $\tan\theta$ repeats itself is 180°, and that is what is meant by the period. Thus $\tan\theta$ has a period of 180°, or $\pi$ radians. So also has $\cot\theta$.

Sec $\theta$ and cosec $\theta$ are both periodic with period $2\pi$ radians.

### Functions of the type $a\sin p\theta$ or $a\sin\omega t$ or $a\sin 2\pi ft$. Oscillations

The graph of $y = a\sin\theta$, where $a$ is any fixed number, is obtained by drawing the graph of $y = \sin\theta$ and then multiplying each ordinate by the same number $a$. This is equivalent merely to altering the scale for $y$.

We could also obtain the graph of $y = a\sin\theta$ by the same construction as in Fig. 188, but using a crank $OP$ of length $a$ units instead of 1 unit.

Now let us see what the graph of $y = \sin 2\theta$ is like. This is the same as $y = \sin\theta$ except that $2\theta$ takes the place of $\theta$ previously. Since the graph of $y = \sin\theta$ crosses the axis when $\theta = \ldots, -180°, 0°, 180°, 360°, \ldots$, the graph of $y = \sin 2\theta$

will cross the axis when $2\theta$ has those values, i.e. when $\theta = \ldots, -90°, 0°, 90°, 180°, \ldots$.

The curve of $\sin\theta$ reaches its maximum and minimum values when $\theta = \ldots, -90°, 90°, 270°, \ldots$, and therefore the curve of $\sin 2\theta$ has its maximum and minimum values when $2\theta = \ldots, -90°, 90°, 270°, \ldots$, i.e. when $\theta = \ldots, -45°, 45°, 135°, \ldots$ Thus the graph of $y=\sin 2\theta$ is the same as that in Fig. 189, except that all the numbers on the $\theta$-axis have now to be halved. This is equivalent to making the scale for $\theta$ twice as large.

The function $\sin\theta$ repeats itself when $\theta$ increases by 360°, and so the function $\sin 2\theta$ repeats itself when $2\theta$ increases by 360°, i.e. when $\theta$ increases by 180°. Thus $\sin 2\theta$ has a period of 180°, or $\pi$ radians. There are *two* complete waves between $\theta=0$ and $\theta=360°$.

In the same way $\sin 3\theta$ is periodic with a period for $\theta$ of $\frac{360°}{3}$, i.e. 120°, or $\frac{2\pi}{3}$ radians. There are three complete waves in the graph between $\theta=0$ and $\theta=360.°$

Generally, if $p$ is any fixed number, the graph of $y=\sin p\theta$ is the same as the graph in Fig. 189, except that the numbers marked there on the $\theta$-axis are now to be read as being the values of $p\theta$ instead of $\theta$, i.e. the graph of $y=\sin p\theta$ is the same as the graph of $y=\sin\theta$, but with a different scale for $\theta$. The function $\sin p\theta$ has a period for $\theta$ of $\frac{360°}{p}$, or $\frac{2\pi}{p}$ radians.

The graph of $y=a\sin p\theta$ is obtained from that of $y=\sin p\theta$ by merely multiplying each ordinate by $a$.

The graph is shown in Fig. 195, angles being marked in radians. If $p$ is a whole number there are $p$ complete waves between 0 and $2\pi$.

If a crank $OP$, of length $a$, rotates with angular velocity $\omega$ radians per sec. starting from the position $OA$ (Fig. 196), then in $t$ secs. it turns through an angle $\omega t$ radians. If $Q$ is the projection of $P$ on the diameter perpendicular to $OA$, then, as $P$ goes round the circle, $Q$ moves along that diameter from $O$ to $B$ then down to $B'$, back to $B$, and so on, i.e. $Q$ oscillates between

$B$ and $B'$. The distance ($y$) of $Q$ above $O$ is equal to $NP$, which is $a \sin \omega t$. When $P$ is in the third or fourth quadrant $\sin \omega t$ is negative and $y$ is negative, indicating that $Q$ is *below* $O$.

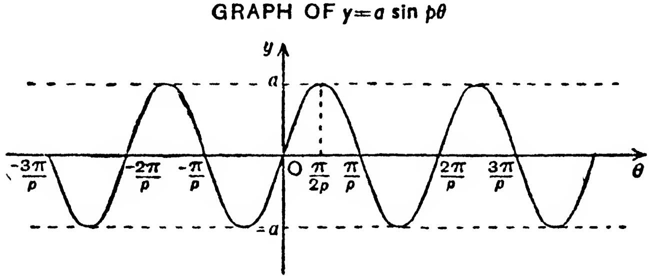



FIG. 195.

If we draw the graph of $y=a \sin \omega t$, marking values of $t$ along the horizontal axis, the ordinate $y$ will give the displacement of $Q$ from $O$ at the time given by the abscissa $t$.

The point $Q$ executes one complete oscillation, i.e. one to-and-fro movement, as for example from $B$ to $B'$ and back to $B$, when the crank makes one complete revolution, i.e. when the angle $\omega t$ increases by $2\pi$ radians, i.e. when $t$ increases by $\frac{2\pi}{\omega}$. The *period of the oscillation* is therefore $\frac{2\pi}{\omega}$ secs. The period is usually denoted by $T$. (Note that the period here is a time, not an angle.)

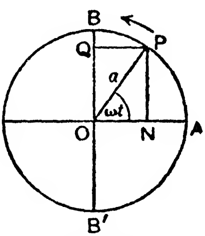



FIG. 196.

The number of oscillations per sec. is called the *frequency* of the oscillations and is usually denoted by $f$ (or sometimes by $n$). It is clear that

$$\text{frequency} = \frac{1}{\text{period}},$$

i.e. $$f = \frac{1}{T} = \frac{\omega}{2\pi}, \text{ and hence } \omega = 2\pi f.$$

The maximum displacement of the moving point ($Q$) from its mean position ($O$) is called the *amplitude* of the oscillations. In Fig. 196 the amplitude is the length of $OB$, i.e. $a$.

Hence the mathematical equation representing oscillations of amplitude $a$ and frequency $f$ is $y=a \sin 2\pi ft$.

**Functions of the type $a \sin (p\theta+\alpha)$ or $a \sin (\omega t+\alpha)$ or $a \sin (2\pi ft+\alpha)$**

Suppose we want to draw the graph of $y=a \sin (\theta+\alpha)$, where $\alpha$ is any fixed angle. This is the same as $y=a \sin \theta$ except that $\theta+\alpha$ takes the place of $\theta$. Now the curve $y=a \sin \theta$ crosses the axis when $\theta= \ldots, -\pi, 0, \pi, 2\pi, \ldots$, and hence the curve $y=a \sin (\theta+\alpha)$ crosses the axis when $\theta+\alpha$ has those values, i.e. when $\theta= \ldots, -\pi-\alpha, -\alpha, \pi-\alpha, 2\pi-\alpha, \ldots$ The points of crossing the axis are therefore all moved through an amount $\alpha$ to the left. The same is true also of any particular ordinate, so that the graph of $y=a \sin (\theta+\alpha)$ is obtained by moving the curve $y=a \sin \theta$ bodily to the left through an amount $\alpha$.

If $\alpha$ is negative it means that the curve is moved to the right instead of to the left; for example, the curve $y=2 \sin (\theta-30°)$ is obtained by drawing the curve $y=2 \sin \theta$ and moving it through 30° to the right.

*Example.*—Sketch the curve $y=4 \sin (\theta+60°)$.

We first draw the curve $y=4 \sin \theta$ which is an ordinary sine curve of amplitude 4. The values of $\theta$ for this curve are marked above the axis. We now have to move the curve through 60° to the left, or, what is equivalent, we can move the $y$-axis through 60° to the right, so that all the numbers on the $\theta$-axis read 60° less than originally. The new values of $\theta$ are marked below the axis. The curve is shown in Fig. 197.

To obtain the graph of $y=a \sin (p\theta+\alpha)$ we write this as $y=a \sin \left\{p\left(\theta+\frac{\alpha}{p}\right)\right\}$. This is the same as $y=a \sin p\theta$ except

that $\theta$ is replaced by $\theta + \dfrac{\alpha}{p}$. The effect of this is to move the curve bodily to the left through an amount $\dfrac{\alpha}{p}$; or, if we prefer, we can move the $y$-axis to the right through an amount $\alpha/p$.

GRAPH OF $y=4 \sin(\theta+60°)$

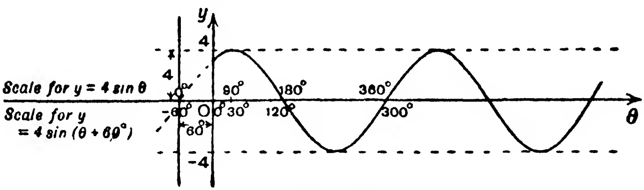



FIG. 197.

*Example.*—Sketch the curve $y=2 \sin(3\theta-150°)$, and find the smallest positive value of $\theta$ for which $y$ is a maximum.

We first draw the curve $y=2\sin 3\theta$, which is a sine wave of amplitude 2 and period $\dfrac{360°}{3}$, i.e. 120°.

GRAPH OF $y = 2 \sin (3\theta - 150°)$

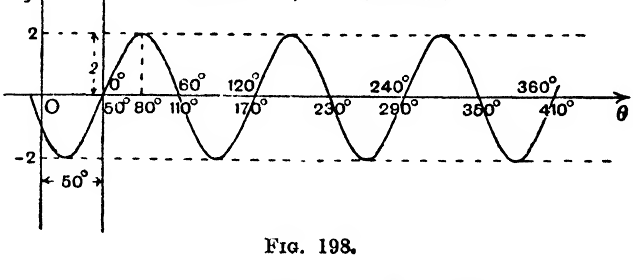



FIG. 198.

There are three waves between 0° and 360° and the curve crosses the axis at 0°, 120°, 240°, 360°, etc. The values of $\theta$ for this curve are marked above the axis.

We now write the required curve as $y=2\sin 3(\theta-50°)$, so that we have to move our original curve through 50° to

the right, or, what is easier, we have to move our $y$-axis through 50° to the left. The values of $\theta$ for the new curve are marked below the axis. The smallest positive value of $\theta$ for which $y$ is a maximum is half way between 50° and 110°, i.e. $\theta = 80°$.

The significance of the constant $\alpha$ can be illustrated by referring again to the rotating vector, or crank, on p. 238. If $OP$ starts from a position $OA'$ (Fig. 199) instead of from $OA$, and if $A\widehat{O}A' = \alpha$ (radians), then, after $t$ secs., $A'\widehat{O}P = \omega t$ and hence $A\widehat{O}P = \omega t + \alpha$ (radians). The displacement of $Q$ from $O$ is therefore given by $y = a \sin(\omega t + \alpha)$.

The period of oscillation of $Q$, which is equal to the time of one revolution of the crank, is not affected by altering the starting position of the crank; the period is still $2\pi/\omega$ secs.

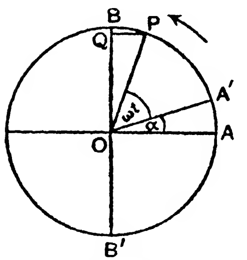



FIG. 199.

The angle $AOP$ between $OP$ and the line of reference $OA$, is called the *phase* at that instant, and $\alpha$ is called the *phase constant* or *phase displacement* (or sometimes the *epoch*). If we imagine two cranks rotating about $O$ in Fig. 199 with the same angular velocity $\omega$, but one starting from $OA$ and the other from $OA'$ at the same instant, the second crank will always be an angle $\alpha$ ahead of the first one, i.e. it "leads" the first one by an amount $\alpha$. For this reason electrical engineers usually say that the oscillation $y = a \sin(\omega t + \alpha)$ has a *lead* of $\alpha$. If $\alpha$ is negative it is called a *lag*.

The second crank is $\alpha/\omega$ secs. ahead of the first in reaching any position; we might call $\alpha/\omega$ the *time lead*.

If we have two oscillations of the same period but different amplitudes, such as

$$y = a \sin(\omega t + \alpha) \text{ and } y = b \sin(\omega t + \beta),$$

the phase of the first at time $t$ is $\omega t + \alpha$, and the phase of the

second at that time is $\omega t + \beta$. The difference of these angles is $\alpha - \beta$; this is called the *phase difference* of the two oscillations. Electrical engineers would say that the first oscillation leads the second by an amount $\alpha - \beta$.

If $\alpha = \beta$ the two oscillations have the same phase at any time; they are said to be *in phase*; if $\alpha$ is not equal to $\beta$ they are said to be out of phase.

It may be noted that, since $a \cos(p\theta + \beta) = a \sin(p\theta + \beta + \pi/2)$, the equation $y = a \cos(p\theta + \beta)$ represents an oscillation of the same amplitude ($a$) and period ($2\pi/p$) as $y = a \sin(p\theta + \beta)$ but leading it by $\pi/2$.

If we draw, with the same axes, the graphs of two oscillations which are in phase with each other, they will cross the time-axis at the same points.

Fig. 200 shows the graphs of two oscillations which are in phase, Fig. 201 the graphs of two oscillations which are out of phase with each other.

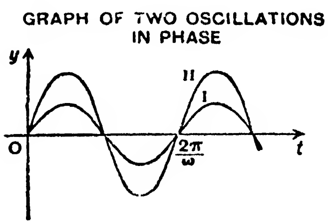



FIG. 200.

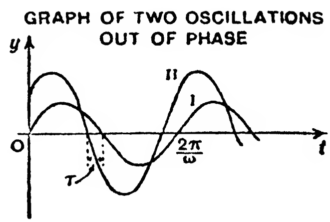



FIG. 201.

In each figure the amplitude of the curve II is twice that of I.

In Fig. 201, the oscillation II leads I by a time $\tau$; for example, the maximum and minimum values of II occur a time $\tau$ before those of I. The angle of lead, or phase difference, is $\omega\tau$.

*Summary.*—The significance of the constants in the equation $y = a \sin(\omega t + \alpha)$ or $y = a \sin(2\pi f t + \alpha)$ is as follows:

$a$ = amplitude; affects only the scale for $y$.

$$\left.\begin{array}{l}\dfrac{2\pi}{\omega}=\dfrac{1}{f}=\text{period}\\[2ex] f=\dfrac{\omega}{2\pi}=\text{frequency}\end{array}\right\}\text{affects only the scale for } t.$$

$\alpha$ = phase displacement; affects only the position of the curve.

### Mechanical and electrical oscillations

Oscillations of the type considered above are of frequent occurrence in mechanics and engineering. They are referred to as sine-wave oscillations, or *sinusoidal* oscillations or *simple harmonic* oscillations. The oscillation may be an actual vibration of a moving point or part of a machine, or it may be merely a convenient way of representing in a formula the value at any time of a quantity which varies between two extreme values $\pm a$ according to a sine law.

Thus in the case of a weight at the end of a spring, and in the motion of the bob of a pendulum (if the amplitude of the oscillations is small), the displacement is represented by an expression of the type above, and in alternating-current theory we meet with voltages given by expressions like $v=V\sin(2\pi ft-\phi)$. In the two former cases there is an actual oscillation—of the weight up and down in a vertical line or of the pendulum-bob in an arc of a circle—but in the third case there is no such motion, though the value of $v$ does oscillate between $\pm V$.

In much the same way every sine-wave (or simple harmonic) oscillation can be pictured as produced by a rotating vector as on pp. 238, 242, though the crank may not be an actuality, but only a geometrical fiction which we use in order to give us a more concrete picture of the behaviour of the quantities with which we are dealing. The electrical engineer uses rotating vectors very frequently in this way.

## Exercise XXX

**1.** Draw an accurate graph of $\sin \theta$ between $\theta = -180°$ and $\theta = +180°$, using a rotating vector as on p. 232. Find, from your graph, the value of $\sin 132°$. What angles are there between $-180°$ and $180°$ having their sine equal to $0{\cdot}71$ ?

State the periods of the following functions of $\theta$ (giving the answers in degrees and in radians).

**2.** (i) $\sin 4\theta$, (ii) $\cos 3\theta$, (iii) $\tan 5\theta$.

**3.** (i) $\sin \frac{\theta}{2}$, (ii) $\sin\left(\frac{3\theta}{2} + \frac{\pi}{6}\right)$, (iii) $\cot p\theta$.

State the frequencies of the following oscillations:

**4.** (i) $y = \sin 100\pi t$, (ii) $y = \sin 40t$, (iii) $y = \cos 2\pi t$.

Write down the amplitude, period and frequency of each of the following oscillations:

**5.** (i) $y = 5 \sin (4t + 1)$, (ii) $v = 230 \sin (50\pi t - 0{\cdot}8)$.

**6.** (i) $s = 3{\cdot}6 \sin\left(\frac{x}{2} + 1{\cdot}4\right)$, (ii) $i = 10 \sin (100\pi t + 6{\cdot}4)$,

(iii) $y = 12 \cos (4t - 2{\cdot}8)$.

**7.** (i) $y = A \sin \frac{px}{q}$, (ii) $y = B \sin\left(\frac{px}{q} + r\right)$.

**8.** Draw the graph of $y = 0{\cdot}5 \sin 2x$ between $x = 0$ and $x = 360°$. From your graph find the value of $\sin 250°$, and compare this with the value obtained by the use of tables.

**9.** Sketch on the same diagram (i.e. with the same axes and the same scales) the graphs of the functions $\sin x$, $\sin 2x$, $\sin \frac{x}{2}$ between $x = 0$ and $x = 2\pi$.

**10.** Draw a rough sketch of the curve $y = 2 \tan 3x$ over two periods commencing at $x = 0$.

**11.** Sketch roughly the curves represented by the expressions in Questions 2–7, showing at least two complete waves in each case, and mark in your figure the amplitude, period and phase displacement, and the points where the curve crosses the axis.

**12.** If a simple pendulum of length $l$ is pulled away from the vertical through a small angle $\theta_0$ and then allowed to fall, the angle $\theta$ which it makes with the vertical at time $t$ later is given

by the formula $\theta = \theta_0 \cos \sqrt{\frac{g}{l}}\,t$, where $g$ is the "acceleration due to gravity" (i.e. 32·19 ft./sec.$^2$, or 981·2 cm./sec.$^2$). Find (i) the period of oscillation of a pendulum of length 1 ft., (ii) the length of a pendulum which beats seconds (i.e. which swings from left to right, or from right to left, in one sec.).

**13.** A crank $OP$, 3 ft. long, rotates at a uniform speed of 30 revs. per min., starting from a position $OA$. If $Q$ is the projection of $P$ on the line through $O$ perpendicular to $OA$, find the equation for the distance of $Q$ from $O$ at a time $t$ secs. later.

**14.** If the crank in Question 13 starts from some other position and passes through the position $OA$ $\frac{1}{4}$ sec. after the start, find the equation for the distance of $Q$ from $O$ at a time $t$ secs. after the start.

**15.** The voltage in a circuit $t$ secs. after the current is switched on is $200 \sin(314t - 50)$. When is the voltage first zero, and when is it first a maximum?

**16.** What is the phase difference between the oscillations $y = 3 \sin\left(4t + \frac{\pi}{8}\right)$ and $y = 5 \sin\left(4t - \frac{\pi}{4}\right)$? Express it (i) as an angle, (ii) as a fraction of the period, (iii) as a time-lead or time-lag.

**17.** Draw an accurate graph of $y = 2 \sin(5x - 3)$ over one period starting from $x = 0$.

**18.** Write down the equation of the sine curve in Fig. 202.

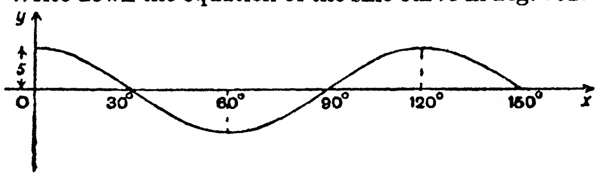



FIG. 202.

**19.** Write down the equation of the sine curve in Fig. 203.

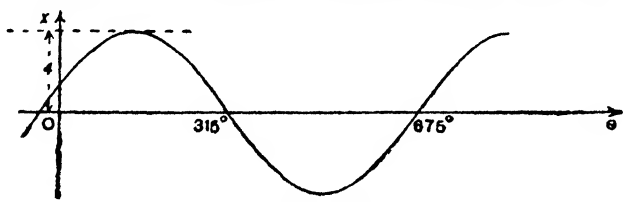



FIG. 203.

**Some other periodic functions and their graphs.**

The sine function is the simplest periodic function, but many other periodic functions occur in practice. For example, in electricity, although the sine wave is the ideal wave form for alternating currents, the voltage produced by a commercial generator is generally of a more complicated wave-type. We shall consider a few periodic functions which can be built up from pure sine functions.

**1. Type $y = c + a \sin (p\theta + \alpha)$**

This is a sine wave in which each ordinate is increased by the same amount $c$. Its graph is therefore obtained by drawing the curve $y = a \sin (p\theta + \alpha)$ and moving it a distance $c$ up the $y$-axis, or, what is equivalent, lowering the $y$-axis through a distance $c$. The period is unaltered; it is still $\frac{2\pi}{p}$.

*Example.*—Fig. 204 shows the graph of $y = 1 + \sin 2\theta$. The value of $y$ oscillates between 0 and 2 (instead of between $-1$ and $+1$).

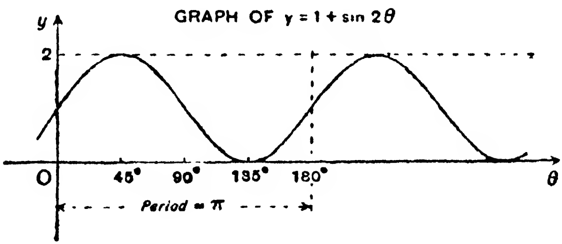



FIG. 204.

**2. Type $y = a \sin \theta + b \cos \theta$**

*Example.*—Draw the graph of $y = 3 \sin \theta + 2 \cos \theta$.

We draw the graph of $3 \sin \theta$ and in the same figure (i.e. using the same axes and the same scales) we draw the graph of $2 \cos \theta$. By means of a pair of dividers or compasses,

we now add the ordinates of the two curves, remembering that a negative ordinate has to be subtracted. The resulting curve, shown by a slightly thicker line, is the graph of $y = 3 \sin \theta + 2 \cos \theta$.

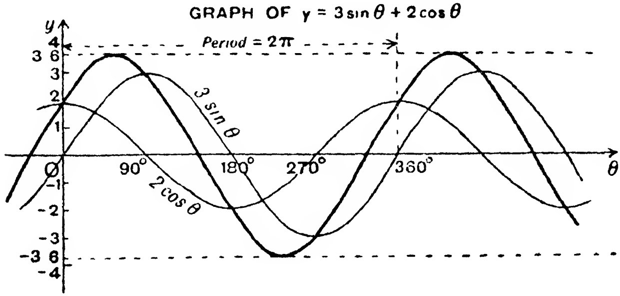



FIG. 205.

It will be noticed that the final curve is periodic, its period being $2\pi$, which is the same as the period of each of the curves from which it is compounded. It can be shown that the final curve is itself a sine wave. In fact we shall show later (in Part III) that if any two sine waves of the same period are added together the resulting curve is itself a sine wave of the same period.

Note that $2 \cos \theta = 2 \sin (\theta + \pi/2)$, so that the component curves in this example are sine waves of the same period, but having a phase difference of $\pi/2$, i.e. a quarter of a period. The amplitude of the resultant curve is seen to be 3·6 (approx.). [Compare p. 292.]

### 3. Type $y = a \sin p\theta + b \sin q\theta$

*Example.*—Draw the graph of $y = 3 \sin \theta + 4 \sin 2\theta$.

Fig. 206 shows the result of adding the ordinates of the curves $y = 3 \sin \theta$ and $y = 4 \sin 2\theta$.

We are here adding two sine waves of different periods.

The resultant curve, shown by a slightly thicker line, is *not* a sine wave, though it *is* a periodic curve.

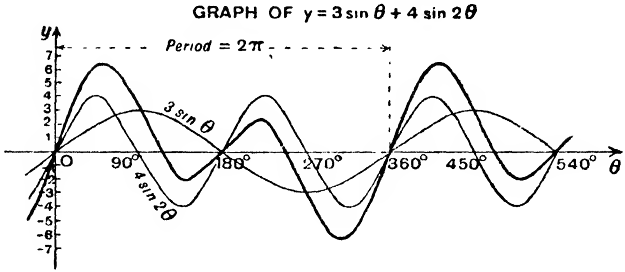



FIG. 206.

*Example.*—Draw the graph of $y=4\sin\theta+\sin 3\theta$.

Fig. 207 shows the result of adding the curves $y=4\sin\theta$ and $y=\sin 3\theta$.

Here again the resultant curve is *not* a sine wave, but it *is* periodic.

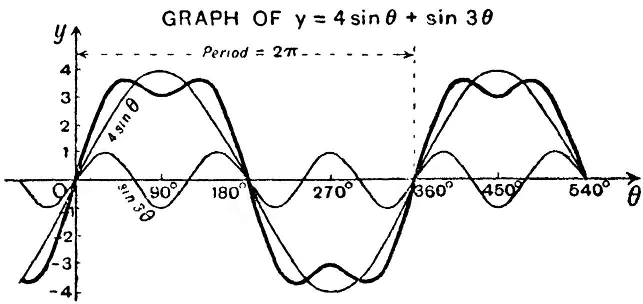



FIG. 207.

**4. Type** $y=a\sin(\theta+\alpha)+b\sin(2\theta+\beta)+c\sin(3\theta+\gamma)+\ \ldots$

This consists of adding a number of sine waves, the frequencies of the second, third, etc., terms on the right-hand side being twice, three times, etc., that of the first term. The component terms on the right-hand side are called, respectively, the first harmonic (or "fundamental"), second harmonic, third harmonic, etc. In the case of a musical note they

represent the fundamental tone and the harmonics or overtones. Compound waves of this type are of importance in many branches of physics and engineering.

The curve in Fig. 207 is more or less typical of waves of this type in which only odd harmonics occur (in the example shown only the first and third harmonics are present); such waves occur frequently in the study of alternating currents. In the curve in Fig. 206 there are only first and second harmonics; this type of wave occurs in considering valve motions.

**Period and frequency of compound wave**

When two sine waves of different periods are compounded (i.e. added together) we can find the period of the resultant wave (which is *not* a sine wave) without drawing a graph. The period of the resultant wave is in fact the least common multiple of the periods of the component waves, as the following example will show.

*Example.*—What is the period of the function $\sin 2x + \sin 4x$?

The period of $\sin 2x$ is $\frac{2\pi}{2}$, i.e. $\pi$ radians, so that the function $\sin 2x$ returns to its original value when $x$ increases by $\pi$ or $2\pi$ or $3\pi$, etc., radians.

The period of $\sin 4x$ is $\frac{2\pi}{4}$, i.e. $\frac{\pi}{2}$ radians, so that $\sin 4x$ returns to its original value when $x$ increases by $\frac{\pi}{2}$ or $\pi$ or $\frac{3\pi}{2}$ or $2\pi$, etc., radians.

The smallest increase in $x$ for which *both* functions, and therefore also their sum, return to their original values is $\pi$, which is the L.C.M. of $\pi$ and $\frac{\pi}{2}$.

The period of the function $\sin 2x + \sin 4x$ is therefore $\pi$ radians.

The same method applies if we compound more than two sine waves; we merely find the L.C.M. of the periods of all the components.

If we are given the frequencies instead of the periods we may first convert to periods, find their L.C.M. and then convert back to frequencies.

*Example.*—Two waves of frequencies 40 and 100 are compounded. What is the frequency of the compound wave?

The periods of the component waves are $\frac{1}{40}$ and $\frac{1}{100}$ sec. The L.C.M. of these is $\frac{1}{20}$ sec. This is the period of the compound wave, and its frequency is therefore 20.

In this example we notice that 20 is the H.C.F. of 40 and 100, and it is not difficult to show that the frequency of any compound wave is the H.C.F. of the component frequencies.

## Exercise XXXI

1. A weight at the end of a spring is vibrating in a vertical line. The length, $x$ in., of the spring at time $t$ secs. is given by $x=6+2\cos 4\pi t$.

Draw a graph of $x$ against $t$. What are the greatest and least lengths of the spring during the motion?

2. Draw the graph of $y=\sin x+\frac{1}{2}\sin 2x$ between $x=0$ and $x=4\pi$.

3. The current in an inductive circuit is given by:

$$i=5\sin 40t-0{\cdot}2\cos 40t.$$

Plot a graph of $i$ against $t$ over one period.

4. In a reciprocating engine the velocity ($v$) of the piston is given by $v=\omega r\left(\sin\theta+\frac{r}{2l}\sin 2\theta\right)$, where $r$ is the length of the crank, $l$ the length of the connecting-rod and $\omega$ the angular velocity of the crank.

If $r=2$, $l=10$ and $\omega=6\pi$, draw a graph of $v$ against $\theta$, from $\theta=0$ to $\theta=4\pi$.

5. The voltage in a cable at time $t$ secs. is equal to

$$200\sin 100\pi t+75\sin 300\pi t.$$

Draw a graph to show how the voltage varies during the first $\frac{1}{50}$ sec.

6. Plot the curve $y = \sin(x - 40°) + 3\sin(3x + 20°)$ from $x = 0°$ to $x = 360°$.

7. Plot the graph of $y = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta$ from $\theta = 0$ to $\theta = 2\pi$.

8. Plot the graph of $d$ against $\theta$ in Question 23 of Exercise XXIX (p. 231).

Find, without drawing a graph, the periods of the following periodic functions of $x$:

9. $\text{Sin } 2x + 3\sin 4x$.

10. $\text{Sin } 2x + 2\sin 3x$.

11. $4\sin\frac{1}{2}x - \sin\frac{1}{3}x$.

12. $\text{Sin } 3\pi x + \cos 5\pi x$.

13. $\text{Sin}\left(3x + \frac{\pi}{4}\right) + 2\sin\left(4x - \frac{\pi}{8}\right)$.

14. $\text{Sin } x + \sin 2x + \sin 3x$.

Simple harmonic oscillations of the following frequencies are compounded. What is the resultant frequency in each case?

15. 50, 200.

16. 30, 45.

17. 225, 250.

18. $f$, $nf$, where $n$ is a whole number. Express your conclusion in this case in words.

## CHAPTER XI

## TRIGONOMETRIC EQUATIONS

### Graphical solution of equations

We have had examples in algebra of solving equations by means of graphs. We can also solve equations in trigonometry by graphical methods.

*Example.*—Solve the equation $x = 2\sin x$ for $x$.

[NOTE.—As explained on p. 230, $\sin x$ is understood to mean $\sin(x \text{ radians})$.]

If we draw the graphs of $y = x$ and $y = 2\sin x$ on the same axes and with the same scales, then at the points where the curves intersect, the ordinates for the two curves will be equal, i.e. $x = 2\sin x$. Hence the roots of the equation are the values of $x$ at the points where the curves intersect.

GRAPHICAL SOLUTION OF EQUATION $x = 2\sin x$

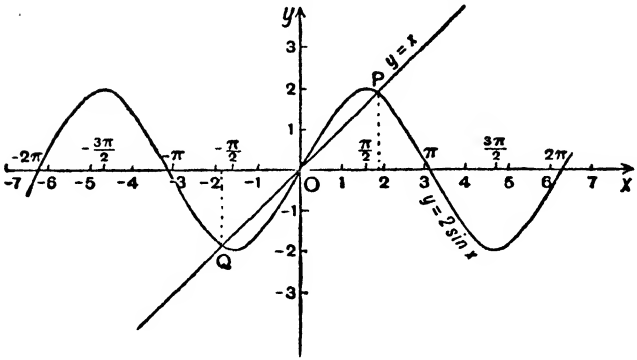



Fig. 208.

The graphs of $y=x$ and $y=2\sin x$ are shown in Fig. 208. [he $x$-axis is scaled (underneath the axis) at unit intervals, ›ut the most convenient values to take for graphing the sine ›urve are $\pi/2$, $\pi$, $3\pi/2$, etc.; these values are marked above he axis. The curves intersect at three points, viz. $O$, $P$ and $Q$. t is evident that they will not intersect again however far ve continue the graphs to the right or left. The abscissæ of the ›oints $O$, $P$ and $Q$ are 0, 1·9 and $-1{\cdot}9$, approx. The equation herefore has three roots, viz. $x=0$ and $x=\pm 1{\cdot}9$ (approx.).

We could get a closer approximation to the roots by drawing he parts of the graph near $P$ and $Q$ on a larger scale. We ıeed only enlarge one portion, say that near $P$, since the root ›t $Q$ is clearly the negative of that at $P$. Fig. 209 shows the ›ortion between $x=1{\cdot}8$ and $x=2{\cdot}0$ on a larger scale.

:able of values for 2 sin ($x$ radians) between $x=1{\cdot}8$ and $x=2$.

| | | | | | |
|---|---|---|---|---|---|
| : .. .. | 1·8 | 1·85 | 1·9 | 1·95 | 2·0 |
| ⅼngle in degrees | 103·14° | 106·00° | 108·87° | 111·73° | 114·60° |
| in $x$ .. | 0·9738 | 0·9613 | 0·9462 | 0·9289 | 0·9092 |
| ; sin $x$ .. | 1·9476 | 1·9226 | 1·8924 | 1·8578 | 1·8184 |

[Note.—1 radian $\simeq 57{\cdot}3° = 57°\ 18'$.]

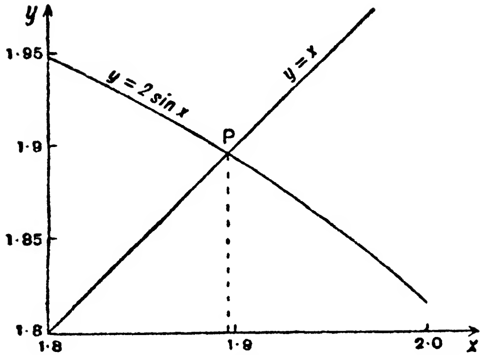



FIG. 209.

From Fig. 209 a more accurate value of the $x$ at $P$ is 1·896. The roots of the equation are therefore 0 and $\pm 1{\cdot}896$ (approx.).

[To test in the equation :

$$\begin{aligned}\sin(1{\cdot}896 \text{ radians}) &= \sin 108^\circ\ 38' \\ &= \sin(180^\circ - 108^\circ\ 38') \\ &= \sin 71^\circ\ 22' = 0{\cdot}9476.\end{aligned}$$

$\therefore$ When $x = 1{\cdot}896$, $2\sin x = 2 \times 0{\cdot}9476 = 1{\cdot}8952 = 1{\cdot}895$ to three places of decimals.]

It is often advisable to rewrite the equation in a different form so as to make the graphs which have to be drawn as simple as possible.

For example, if we wish to solve the equation $x^2 \tan x = 1$, instead of finding the intersections of the graphs of $y = x^2 \tan x$ and $y = 1$, we might write the equation in the form $x^2 = \cot x$ (by dividing both sides of the equation by $\tan x$), and find the intersections of the graphs of $y = x^2$ and $y = \cot x$, which are curves of well-known shape. Alternatively we could put the equation into the form $\tan x = \frac{1}{x^2}$, and find the intersections of the graphs of $y = \tan x$ and $y = \frac{1}{x^2}$.

**Angles having a given sine, cosine or tangent, etc.**

To find the angles whose sines are equal to $\frac{1}{2}$ is equivalent to solving the equation $\sin\theta = \frac{1}{2}$ for $\theta$. We could do this graphically by drawing the graph of $y = \sin\theta$ and finding where it cuts the line $y = \frac{1}{2}$.

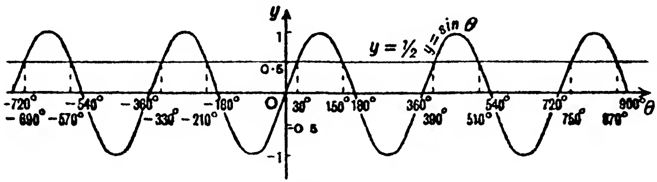



FIG. 210.

From Fig. 210 it is seen that the line $y = \frac{1}{2}$ cuts the curve $y = \sin\theta$ in an infinite number of points. The values of $\theta$ at those points are . . ., $-690°$, $-570°$, $-330°$, $-210°$, $30°$, $150°$, $390°$, $510°$, $750°$, $870°$, . . .

Similarly, if we take any other value between $-1$ and $+1$, call it $k$, we see that the line $y = k$ cuts the curve $y = \sin\theta$ in an infinite number of points; that is, the equation $\sin\theta = k$ has an infinite number of roots.

Since the curve $y = \sin\theta$ repeats itself every $360°$ we need only to find the roots between $\theta = 0°$ and $\theta = 360°$ or the roots which lie within any other period, say between $-180°$ and $+180°$. If we then add or subtract multiples of $360°$, we shall obtain all the other roots.

The same is true if we want to find all the angles having a given cosine, or tangent, or cotangent, secant or cosecant.

We can find the roots between $0°$ and $360°$ without drawing a graph, as the following examples will show.

*Example.*—Find all the values of $\theta$ for which $\sin\theta = \frac{1}{2}$.

First find the angles between $0°$ and $360°$. Since $\sin\theta$ is positive, $\theta$ must lie in the first or second quadrant.

The angle in the first quadrant whose sine is $\frac{1}{2}$ is 30°. The angle in the second quadrant is $180° - 30°$, i.e. 150° (Fig. 211).

All the other values are found by adding or subtracting multiples of 360°. Thus we get

$$\ldots, -690°, -330°, 30°, 390°, 750°, \ldots$$

and $\quad \ldots, -570°, -210°, 150°, 510°, 870°, \ldots$

the dots indicating that we can continue indefinitely in each direction.

All these angles are included in the general formula

$$\theta = 30° \pm n \,.\, 360° \text{ or } 150° \pm n \,.\, 360°,$$

where $n$ denotes *any* integer.

*Example.*—Find $\theta$ if $\cos\theta = -0{\cdot}6202$.

First to find the angles between 0° and 360° which satisfy this equation. Since $\cos\theta$ is negative, $\theta$ must lie in the second or third quadrant.

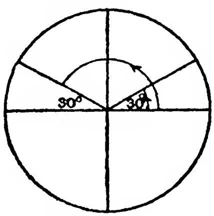



FIG. 211.

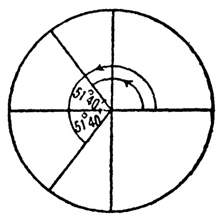



FIG. 212.

The acute angle whose cosine is $0{\cdot}6202$ is (from tables) 51° 40′. The required angles (shown in Fig. 212) are therefore $180° - 51°\ 40'$, i.e. 128° 20′, and $180° + 51°\ 40'$, i.e. 231° 40′. All the other angles are obtained by adding or subtracting multiples of 360°; they are

$\ldots, -591°\ 40', -231°\ 40', 128°\ 20', 488°\ 20', 848°\ 20', \ldots$

and

$\ldots, -488°\ 20', -128°\ 20', 231°\ 40', 591°\ 40', 951°\ 40', \ldots$

or, more concisely,

$$128°\ 20' \pm n \,.\, 360° \text{ or } 231°\ 40' \pm n \,.\, 360°,$$

where $n$ is any integer.

It has already been observed (p. 227) that each of the six trigonometric ratios is positive in two quadrants and negative in the other two. Thus, whichever of the six ratios is given, there will be two angles between 0° and 360° which have the given ratio.

Thus we have the following *rule* :

*To find all the angles having a given sine, cosine, tangent, etc., first write down the two angles between* 0° *and* 360° *having the given ratio, and then add or subtract multiples of* 360°.

In the case of the tangent we can write down the angles still more easily. If we refer to the graph of $\tan\theta$ on p. 235, we see that all the angles having a given tangent differ by multiples of 180°; so that we need only to find *one* angle having the given tangent and then add or subtract multiples of 180°.

For example, all the angles whose tangents are equal to 1 are included in the formula $45° \pm n\,.\,180°$, where $n$ is any integer.

Similarly, the general solution of the equation $\tan\theta = \tan\alpha$ is $\theta = \alpha \pm n\,.\,180°$.

The notation $\sin^{-1} x$ is often used to denote an angle whose sine is equal to $x$, with similar meanings for $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, etc. Thus in the last example we have found all the values of $\cos^{-1}(-0{\cdot}6202)$. The student should be careful to remember that $\sin^{-1} x$ does *not* mean $\sin x$ raised to the power $-1$.* To avoid ambiguity it is usual to write $\sin x$ to the power $-1$ as $(\sin x)^{-1}$.

*Example.*—Find all the angles $\theta$ between 0° and 360° which satisfy the equation $5\sin(2\theta - 45°) = 1{\cdot}47$.

$$\sin(2\theta - 45°) = \frac{1{\cdot}47}{5} = 0{\cdot}294$$

$$\therefore\ 2\theta - 45° = \sin^{-1} 0{\cdot}294.$$

* In Continental and American books the notation arc sin $x$, arc cos $x$, etc., is used in place of $\sin^{-1} x$, $\cos^{-1} x$, etc.

9

The two values of $\sin^{-1} 0{\cdot}294$ between $0°$ and $360°$ are $17° \, 6'$ and $162° \, 54'$.

$$\therefore\ 2\theta - 45° = 17° \, 6' \pm n \,.\, 360°$$
$$\text{or } 162° \, 54' \pm n \,.\, 360°, \text{ where } n \text{ is any integer.}$$
$$\therefore\ 2\theta = 62° \, 6' \pm n \,.\, 360°$$
$$\text{or } 207° \, 54' \pm n \,.\, 360°$$
$$\therefore\ \theta = 31° \, 3' \pm n \,.\, 180°$$
$$\text{or } 103° \, 57' \pm n \,.\, 180°, \text{ where } n \text{ is any integer.}$$
$$\text{i.e. } \theta = \ldots, -148° \, 57', 31° \, 3', 211° \, 3', 391° \, 3', \ldots$$
$$\text{or } \ldots, -76° \, 3', 103° \, 57', 283° \, 57', 463° \, 57', \ldots$$

We have to pick out the values of $\theta$ between $0°$ and $360°$; they are

$$31° \, 3', 211° \, 3' \text{ and } 103° \, 57', 283° \, 57';$$

or, arranged in order,

$$31° \, 3', 103° \, 57', 211° \, 3' \text{ and } 283° \, 57'.$$

**To find an angle when its sine and cosine are given**

If an angle is known to be acute we can find the angle (from tables) if we are given *either* its sine *or* its cosine (or its tangent or cotangent, etc.).

If an angle is known to lie between $0°$ and $360°$ and the sine is given, there are two possible values for the angle: e.g. if the sine is $\frac{1}{2}$, the angle may be $30°$ or $150°$. Similarly, if the cosine is given there are two possible values. If, however, we are given both the sine *and* the cosine then the angle is determined.

For example, suppose we are given that $\sin\theta = \frac{1}{2}$ and $\cos\theta = -\frac{\sqrt{3}}{2}$ and that $\theta$ lies between $0°$ and $360°$. Then, since $\sin\theta = \frac{1}{2}$, $\theta$ must be either $30°$ or $150°$; also, since $\cos\theta = -\frac{\sqrt{3}}{2}$, $\theta$ must be either $150°$ or $210°$. The only possible value for $\theta$, therefore, is $150°$.

The quickest way to find the angle is to use the "*all, sin, tan, cos* rule" (p. 227). Thus, in the above example, since $\sin\theta$ is positive, $\theta$ lies in the 1st or 2nd quadrant; since $\cos\theta$

is negative, $\theta$ lies in 2nd or 3rd quadrant. The only possible quadrant therefore is the 2nd. The value of the angle can now be found either from its sine *or* from its cosine.

*Example.*—Find $\theta$ if $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$, assuming that $\theta$ lies between 0° and 360°.

Since $\sin\theta$ is $-ve$, $\theta$ lies in the 3rd or 4th quadrant.

Since $\cos\theta$ is $+ve$, $\theta$ lies in the 1st or 4th quadrant.

Hence $\theta$ must lie in the 4th quadrant.

From a table of sines we find that the acute angle whose sine is $\frac{4}{5}$, i.e. 0·8, is 53° 8′.

Hence $\theta = 360° - 53°\ 8'$
$= 306°\ 52'$.

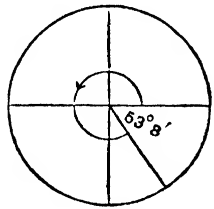



FIG. 213.

## Exercise XXXII

[Exercises 1–7 are to be solved *graphically*.]

1. Solve the equation $\cos(x \text{ radians}) = 1 - \frac{x}{2}$.

2. Find the smallest positive root of the equation $x\tan x = 1$, the angle being in radians.

3. Find the roots of the equation $\sin(x+25)° = \cos 2x°$ which lie between 0 and 180.

4. When a crossed belt of length $l$ in. passes over two pulley wheels of radii $r_1$ and $r_2$ in., the distance $d$ in. between their centres is given by $d = (r_1 + r_2)\operatorname{cosec}\theta$, where $\theta$ is determined from the equation

$$l = 2(r_1 + r_2)\left(\frac{\pi}{2} + \theta + \cot\theta\right) \text{ (see p. 201).}$$

If $l = 80$, $r_1 = 6$ and $r_2 = 4$, find $\theta$ and hence $d$.

5. The force on a piston is proportional to $\cos\theta + \frac{r}{l}\cos 2\theta$, where $r$ and $l$ are the lengths of the crank and connecting-rod. If $l = 4r$, find the smallest positive value of $\theta$ for which the force on the piston is zero.

6. Fig. 214 shows a cross-section of a gutter full of water. The area of the cross-section of the water is $\frac{1}{2}r^2(\theta - \sin\theta)$ sq. in. If $r=3$ and the area is $4\frac{1}{2}$ sq. in., find the value of $\theta$, and express the angle in degrees.

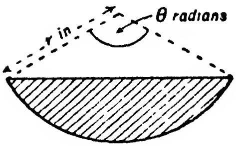



FIG. 214.

7. In a certain type of ammeter the current, $i$ amps., when the deflection of the needle is $\theta°$ is given by

$$i = 0{\cdot}735\sqrt{\frac{\theta}{\sin(\theta+30)°}}.$$

Find the deflection when a current of 3 amps. passes through the instrument.

8. In which quadrant does an angle lie if its sine and cosine are (i) both positive, (ii) both negative ?

Find the two values between 0° and 360° of the following:

9. $\text{Sin}^{-1}\, 0{\cdot}33$.
10. $\text{Cos}^{-1}\, 0{\cdot}472$.
11. $\text{Sin}^{-1}\,(-0{\cdot}92)$.
12. $\text{Tan}^{-1}\, 4{\cdot}23$.
13. $\text{Sec}^{-1}\, 1{\cdot}40$.
14. $\text{Cos}^{-1}\,(-0{\cdot}845)$.

Find the values between $-180°$ and 180° of the following:

15. $\text{Cot}^{-1}\, 0{\cdot}708$.
16. $\text{Sin}^{-1}\,(-0{\cdot}265)$.

Find, in radians, the angles between 0 and $2\pi$ which satisfy the following equations:

17. $\text{Sin}\,\theta = \frac{1}{4}$.
18. $\text{Tan}\, x = -1$.
19. $\text{Cos}\,\alpha = 0{\cdot}2$.
20. $\text{Cosec}\,\theta = 2$.

Find all the values of $\theta$ between 0° and 360° which satisfy the following equations:

21. $2\cos 4\theta = 0{\cdot}75$.
22. $3\sin\dfrac{\theta}{2} = 2{\cdot}58$.
23. $\text{Sin}\, 2\theta = -0{\cdot}5$.
24. $5\tan 3\theta = 9{\cdot}325$.

25. The angular displacement $\phi$ degrees of a pendulum at time $t$ sec. is given by $\phi = 15\sin 3\pi t$. Find the times during the first second at which the displacement is $+10°$.

26. The voltage in a circuit $t$ sec. after the current is switched on is $200\sin(314t - 50)$ volts. Find, by calculation, (i) when the voltage is first zero, and (ii) when it first reaches its maximum value.

27. The displacement $s$ in. of a slide-piece in a certain mechanism at time $t$ sec. is given by $s = 6\sin(3t + 0{\cdot}8)$. Find the

smallest positive value of $t$ for which the displacement is (i) $+3$ in., (ii) $-3$ in.

Find the angles $\theta$, between 0° and 360°, which satisfy the following:

28. Sin $\theta = 0{\cdot}81$ and cos $\theta = -0{\cdot}5864$.

29. Sin $\theta = -\frac{5}{13}$ and cos $\theta = \frac{12}{13}$.

30. Sin $\theta = -\frac{8}{17}$ and cos $\theta = -\frac{15}{17}$.

**Trigonometric equations**

*Example.*—Find all the angles between 0° and 360° which satisfy the equation

$$\tan^2\theta - 4\tan\theta + 1 = 0.$$

This is a quadratic equation in tan $\theta$; solving the quadratic in the ordinary way we have:

$$\begin{aligned}\tan\theta &= \frac{4 \pm \sqrt{16-4}}{2}\\ &= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}\\ &= 2 \pm 1{\cdot}7321\\ &= 3{\cdot}7321 \text{ or } 0{\cdot}2679.\end{aligned}$$

When $\tan\theta = 3{\cdot}7321$, $\theta = 75° \pm n\,.\,180°$.

When $\tan\theta = 0{\cdot}2679$, $\theta = 15° \pm n\,.\,180°$.

$\therefore$ The possible values of $\theta$ between 0° and 360° are 15°, 75°, 195° and 255°.

If two or more ratios occur in the equation we must transform the equation into a form in which only one ratio occurs.

*Example.*—Solve the equation

$$\sin^2\theta = \cos\theta.$$

Since $\sin^2\theta = 1 - \cos^2\theta$, the equation can be written in the form

$$1 - \cos^2\theta = \cos\theta,$$

which contains only $\cos\theta$.

$$\therefore\ \cos^2\theta+\cos\theta-1=0.$$

This is a quadratic in $\cos\theta$; solving the quadratic we have

$$\begin{aligned}\cos\theta&=\frac{-1\pm\sqrt{1+4}}{2}\\&=\frac{-1\pm\sqrt{5}}{2}\\&=\frac{-1\pm2{\cdot}23607}{2}\\&=\frac{1{\cdot}23607}{2}\text{ or }-\frac{3{\cdot}23607}{2}\\&=0{\cdot}6180\text{ or }-1{\cdot}6180.\end{aligned}$$

The latter value is impossible, since $\cos\theta$ cannot be less than $-1$ for any real angle $\theta$.

$$\therefore\ \cos\theta=0{\cdot}6180$$

$$\begin{aligned}\therefore\ \theta&=51^\circ\ 50'\pm n\,.\,360^\circ\\&\text{or } 308^\circ\ 10'\pm n\,.\,360^\circ.\end{aligned}$$

*Example.*—Find the angles $x$ between $0^\circ$ and $360^\circ$ which satisfy the equation

$$\sin 2x=\tfrac{1}{2}\cos 2x.$$

Dividing both sides of the equation by $\cos 2x$, we have

$$\tan 2x=\tfrac{1}{2},$$

which contains only one ratio, viz. $\tan 2x$.

$$\therefore\ 2x=26^\circ\ 34'\pm n\,.\,180^\circ,\text{ where } n \text{ is any integer.}$$

$$\therefore\ x=13^\circ\ 17'\pm n\,.\,90^\circ,\text{ where } n \text{ is any integer.}$$

The values of $x$ between $0^\circ$ and $360^\circ$ are

$$13^\circ\ 17',\ 103^\circ\ 17',\ 193^\circ\ 17',\ 283^\circ\ 17'.$$

## Exercise XXXIII

Find the angles between 0° and 360° which satisfy the following equations :

1. $4 \sin^2 \theta = 1$.

2. Tan $x = 5 \cot x$. $\left[\text{Hint.—Write } \cot x \text{ as } \frac{1}{\tan x}.\right]$

3. $\text{Sin}^2 \theta - 2 \cos^2 \theta + 1 = 0$.

4. $\text{Cos}^2 x = 3 (1 + \sin x)$.

5. Tan $x + 2 \cot x - 3 = 0$ [see hint to Question 2].

6. $\text{Sec}^2 \theta = \frac{1}{2} \tan \theta + 1$.

7. $\text{Cos}^2 \frac{x}{2} = 0{\cdot}4$.

8. Sin $3x = \cos 3x$.

9. Sin $x + \tan x = 0$.

10. The angular velocity of a connecting-rod in an engine is

$$\frac{\omega \cos \theta}{\sqrt{\left(\frac{l^2}{r^2} - \sin^2 \theta\right)}}.$$

If $r = 1{\cdot}4$ and $l = 7$, find the values of $\theta$ for which the angular velocity is $\frac{1}{8}\omega$.

11. Barr's formula for the efficiency of a worm-gearing is

$$e = \frac{\tan \alpha \, (1 - \mu \tan \alpha)}{\mu + \tan \alpha}.$$

If $e = 0{\cdot}93$ and $\mu = 0{\cdot}02$ find the angle $\alpha$. (Only the smallest positive value is required here.)

12. Fig. 215 shows a cam which rotates about $O$ and imparts a vertical motion to the follower $F$, the follower moving up and down in a vertical line through $O$. Show that, when the point of contact $P$ lies on the arc $AB$, the height ($h$) of $P$ above $O$ is given by

$$h = d \cos \theta + \sqrt{b^2 - d^2 \sin^2 \theta}.$$

If $b = \frac{1}{2}$ in. and $d = 2\frac{3}{4}$ in., find the value of $\theta$ when $h = 3{\cdot}1$ in.

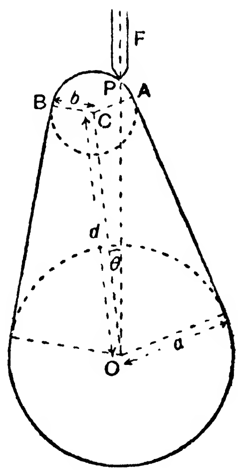



FIG. 215.

**Some useful identities**

The relation $\sin^2\theta+\cos^2\theta\equiv 1$ was proved earlier for any *acute* angle $\theta$, but it is easily proved to be true for any angle $\theta$ whatever its magnitude, positive or negative.

For in Fig. 178, $ON^2+NP^2=OP^2$ in whatever quadrant $OP$ lies.

$$\therefore \left(\frac{ON}{OP}\right)^2+\left(\frac{NP}{OP}\right)^2=1.$$

But $\dfrac{ON}{OP}=\pm\cos\theta$ and $\dfrac{NP}{OP}=\pm\sin\theta$, the signs to be taken depending on the quadrant in which $OP$ lies. The $-$ signs disappear when we square them and thus in any case we have

$$\cos^2\theta+\sin^2\theta\equiv 1.$$

So also the relations

$$1+\tan^2\theta\equiv\sec^2\theta \text{ and } 1+\cot^2\theta\equiv\operatorname{cosec}^2\theta$$

are true for all values of $\theta$.

It can be proved also that the relations :

$$\sin(90^\circ-\theta)=\cos\theta, \qquad \cos(90^\circ-\theta)=\sin\theta,$$
$$\tan(90^\circ-\theta)=\cot\theta, \qquad \cot(90^\circ-\theta)=\tan\theta,$$
$$\sec(90^\circ-\theta)=\operatorname{cosec}\theta, \qquad \operatorname{cosec}(90^\circ-\theta)=\sec\theta,$$

are true, whatever the angle $\theta$, by drawing figures in which the angle lies in different quadrants.

*Example.*—Verify the relation $\sin(90^\circ-\theta)=\cos\theta$ when $\theta=120^\circ$.

$$\begin{aligned}\sin(90^\circ-120^\circ)&=\sin(-30^\circ)\\ &=-\sin 30^\circ \text{ (Fig. 216)}\\ &=-\tfrac{1}{2}\\ \cos 120^\circ&=-\cos 60^\circ=-\tfrac{1}{2}.\end{aligned}$$

Hence the relation is true when $\theta=120^\circ$.

If $\theta$ is an acute angle, $180^\circ-\theta$ is an angle in the second quadrant and, from the rule on p. 229, we have from Fig. 217 that

$$\sin(180^\circ-\theta)=\sin\theta, \quad \cos(180^\circ-\theta)=-\cos\theta,$$
$$\tan(180^\circ-\theta)=-\tan\theta.$$

Also if $\theta$ is acute, $(-\theta)$ is in the fourth quadrant and therefore, from Fig. 217,

$$\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta.$$

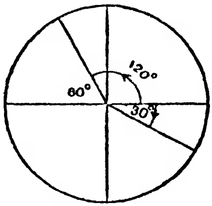



FIG. 216.

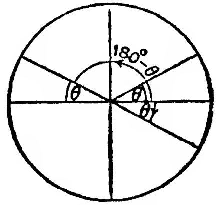



FIG. 217.

These six relations can also be proved to be true whatever the magnitude of the angle $\theta$, positive or negative.

*Example.*—Verify that $\cos(180° - \theta) = -\cos\theta$ when $\theta = -110°$.

$$\begin{aligned} \cos\{180° - (-110°)\} &= \cos(180° + 110°) = \cos 290° \\ &= \cos 70°, \text{ from Fig. 218.} \\ -\{\cos(-110°)\} &= -(-\cos 70°), \text{ from Fig. 219} \\ &= \cos 70°. \end{aligned}$$

Hence the relation is true when $\theta = -110°$.

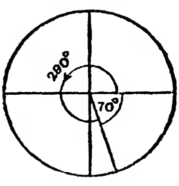



FIG. 218.

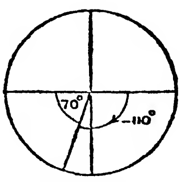



FIG. 219.

The following identical relations can also be proved to be true for all angles $\theta$:

$$\sin(90° + \theta) = \cos\theta, \quad \cos(90° + \theta) = -\sin\theta,$$
$$\tan(90° + \theta) = -\cot\theta;$$

$$\sin(180° + \theta) = -\sin\theta, \quad \cos(180° + \theta) = -\cos\theta,$$
$$\tan(180° + \theta) = \tan\theta.$$

We shall prove only the first ; the proofs of the remainder can be left as an exercise for the student.

Sin $(90° + \theta)$

$= \sin\{90° - (-\theta)\} = \cos(-\theta)$ from the relations on p. 264

$= \cos\theta.$

The relations between the ratios of an angle $\theta$ and the ratios of the angles $(-\theta)$, $180° - \theta$ and $180° + \theta$ are illustrated on the graphs of the ratios.

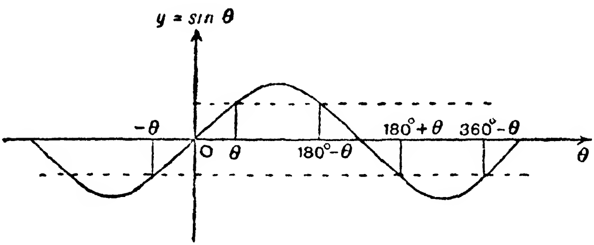



Fig. 220.

For example, the graph of $y = \sin\theta$ in Fig. 220 shows that

$$\sin(180° - \theta) = \sin\theta,$$

and that $\qquad \sin(180° + \theta) = \sin(-\theta) = -\sin\theta.$

[In Fig. 220, we have marked $\theta$ as an angle between 0° and 90° ; the student is advised to take other values for $\theta$ on the graph and to verify these relations for the values he chooses.]

The student is not expected to remember all these relations. In practice, if we want any one of them we draw a rough figure in which $\theta$ is *acute*, and read off the relation from the figure. We then know that it is true for all values of $\theta$.

This same remark applies to expressions of the form

the stated scale. To distinguish between the two opposite directions (or "senses") on a line we often put an arrowhead on the line.

We usually speak of "the vector $\overrightarrow{AB}$" when we mean the vector represented by a line $AB$ (in the sense *from $A$ to $B$*) in a vector diagram. $A$ and $B$ are called the *initial* and *final* points of the vector respectively. If the sense of the vector were from $B$ to $A$, so that the arrowhead would be reversed in Fig. 221, we should write it as "the vector $\overrightarrow{BA}$."

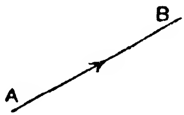


FIG. 221.

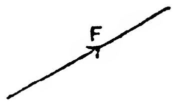


FIG. 222.

We sometimes denote a vector by a single letter with an arrow over it; thus, if the length of the vector in Fig. 222 be $F$, we should speak of "the vector $\vec{F}$." The arrow over the $F$ is necessary to distinguish between a vector and a number or length. Many books use a heavier type instead of an arrow, and write the vector as $\mathbf{F}$; electrical engineers use a dot and write it as $\underset{\bullet}{F}$.

Another useful way of writing a vector is to take some standard direction, such as $OX$ in Figs. 223 and 224, from which to measure angles. Then if the angle between $OX$ and the direction of $\vec{F}$ is $\alpha$, we write the vector $\vec{F}$ as $F_\alpha$; electrical engineers often write it as $F\angle\alpha$.

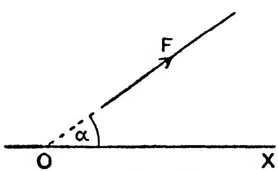


FIG. 223.

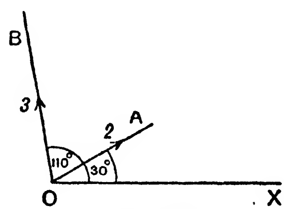


FIG. 224.

For example, in Fig. 224, $\overrightarrow{OA}=2_{30°}$, $\overrightarrow{OB}=3_{110°}$, referred to $OX$ as direction of reference.

[NOTE.—It should be carefully observed that when we speak of the direction $OX$ we mean the direction *from $O$ to $X$*. The easiest way to make quite sure that we label the angle correctly in the diagram is to take a pen and lay it along $OX$ with the nib pointing towards $X$ and then turn the pen about $O$ (in an anti-clockwise direction) until it falls along the vector with the nib pointing in the direction of the arrow. The angle through which the pen has turned is the angle $\alpha$.

For example, vector $\overrightarrow{EO}$ in Fig. 225 is $2_{240°}$, *not* $2_{60°}$.]

The *resultant* of two vectors is defined by the "parallelogram law"; that is, the resultant of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Fig. 226 is the vector $\overrightarrow{OC}$, where $OC$ is the diagonal through $O$ of the parallelogram formed on the sides $OA$ and $OB$.

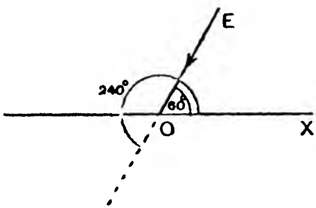



FIG. 225.

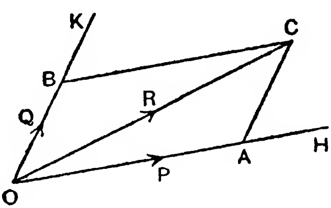



FIG. 226.

This is the law by which, in mechanics, we compound two forces acting at a point or two velocities.

The vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are the *components* of the vector $\overrightarrow{OC}$ in the directions $OH$ and $OK$. The components of a vector in any two directions can be found by reversing the construction for the resultant, that is by drawing lines through $C$ parallel to the given directions cutting them in $A$ and $B$.

It is frequently necessary to find the components of a vector in two directions which are at right angles, such as $OX$, $OY$ in Fig. 227. In that case the magnitudes of the components are called the *resolved parts* or *resolutes* of the vector

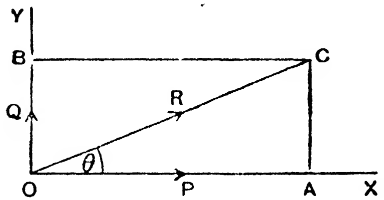



FIG. 227.

in those directions. If the resolved parts along $OX$, $OY$ are $P$ and $Q$ and the magnitude of the resultant is $R$, and if the direction of the resultant makes an acute angle $\theta$ with $OX$, then, from Fig. 227,

$$P = R\cos\theta, \quad Q = R\sin\theta;$$

Also $$R = \sqrt{P^2 + Q^2}, \quad \tan\theta = \frac{Q}{P}.$$

**Resultant of any number of vectors**

In Fig. 226, since $OACB$ is a parallelogram, $AC$ is equal and parallel to $OB$. We can therefore find the resultant $\vec{R}$ (i.e. $\overrightarrow{OC}$) by drawing $OA$ and $AC$, of lengths $P$ and $Q$, parallel to the vectors $\vec{P}$ and $\vec{Q}$. It is not necessary to draw the whole parallelogram. The construction is shown in Fig. 228.

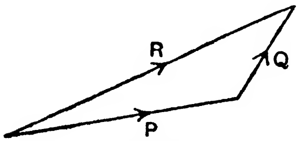



FIG. 228.

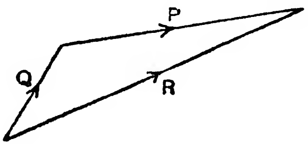



FIG. 229.

We might describe the construction briefly thus: Place the vectors $\vec{P}$, $\vec{Q}$ end to end, so that the initial point of one vector coincides with the final point of the other; the resultant is then represented by the line joining the free ends, in the direction *from* the first point *to* the last.

The order in which we take the vectors makes no difference, as is shown by Fig. 229, which is equivalent to drawing the triangle $OBC$ of Fig. 226.

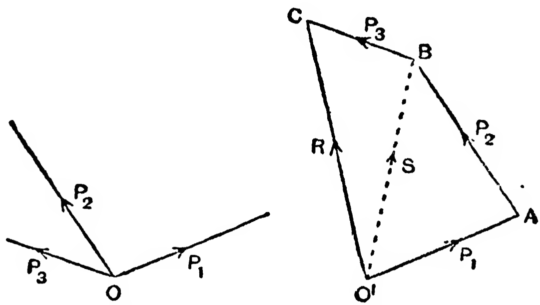



FIG. 230 (*a*). FIG. 230 (*b*).

Now suppose we have three vectors, $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ (Fig. 230 (*a*)). $\vec{P}_1$ and $\vec{P}_2$ have a resultant $\vec{S}$ (Fig. 230 (*b*)). Compounding $\vec{S}$ with $\vec{P}_3$, they have a resultant $\vec{R}$. Thus $\vec{R}$ is the result of compounding the three vectors $\vec{P}_1$, $\vec{P}_2$ and $\vec{P}_3$. The order in which we take the three vectors makes no difference to the final result. The vector $\vec{R}$ is called the resultant of the three given vectors.

If $\vec{P}_1$, $\vec{P}_2$, $\vec{P}_3$ represent three forces acting at $O$, they are equivalent to a single force $\vec{R}$ acting at $O$.

We can find the resultant of any number of vectors by extending the construction above. The rule is: Place the vectors end to end, so that the initial point of each vector

coincides with the final point of the previous one, that is, so that the direction of the arrows as we go round the figure $O'ABC$ is continuous. The resultant is represented by the line joining the free ends in the direction from the first point to the last.

$O'A$, $AB$, $BC$ (in Fig. 230 (*b*)) may be regarded as the links of a broken chain, and $CO'$ as the final link which completes the chain. This link *reversed* is the resultant. Hence the above rule is called the *chain rule*.

*Example.*—Find, graphically, the resultant of the vectors $2{\cdot}5_{30°}$, $4_{70°}$, $2_{125°}$, $6_{212°}$.

Take $OX$ as direction of reference. Taking a scale of 1 unit = 1 cm, draw $OA$ of length 2·5 cm. making an angle 30° with $OX$ (Fig. 231). From $A$ draw $AB$ of length 4 cm., making an angle 70° with $OX$. Similarly, draw $BC$ and $CO$ to represent $2_{125°}$ and $6_{212°}$.

Join $OD$. Then $\overrightarrow{OD}$ is the resultant.

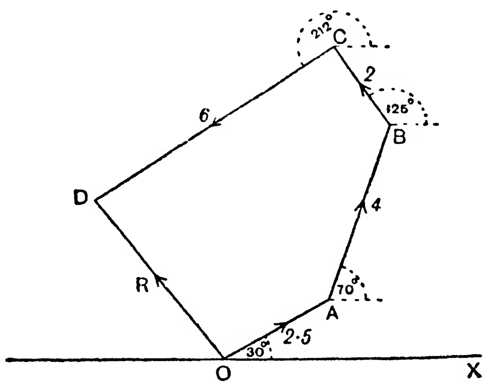



FIG. 231.

By measurement it is found that $R = 4{\cdot}3$ cm. which represents 4·3 units (on our scale) and $X\widehat{O}D = 128°$.

Hence the resultant is the vector $4{\cdot}3_{128°}$.

[In Fig. 231 the scale has been reduced for convenience of printing. The student should use as large a scale as possible, of course.]

*Example.*—A body is pulled simultaneously by forces of 20 lb. wt. due N., 25 lb. wt. due E. and 16 lb. wt. in a direction S.W. Find the direction in which it moves.

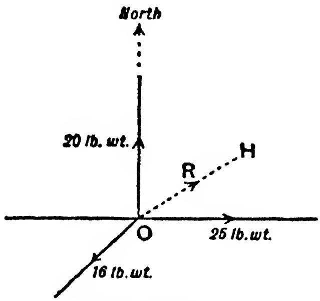



FIG. 232 (*a*).

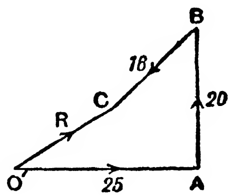



FIG. 232 (*b*).

The forces are shown in Fig. 232 (*a*). Their resultant is found by the chain rule in Fig. 232 (*b*); it is represented by $\overrightarrow{O'C}$.

The body moves in the direction of the resultant force, *i.e.* in the direction $OH$ (which is drawn parallel to $O'C$). By measurement we find that $OH$ makes an angle $56\frac{1}{2}°$ with the northerly direction, i.e. the body moves in the direction N. $56\frac{1}{2}°$ E.

**Resolved part of a vector in any direction**

In Fig. 227 $OA$ is the resolved part of $\overrightarrow{OC}$ in the direction $OX$; it is the projection of $OC$ on that direction.

If $\overrightarrow{OC}$ represents a force, $\overrightarrow{OA}$ may be regarded as the *effective part* of the force in the direction $OX$, since the component $\overrightarrow{OB}$ has no tendency to move the body in the direction $OX$.

Now suppose the angle $\theta$ between $OX$ and $OC$ is an obtuse angle, as in Fig. 233.

The force $\overrightarrow{OC}$ is equivalent to the two forces $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The effective part so far as motion along $OX$ is concerned is $\overrightarrow{OA}$, which is a force of magnitude $P$ in the direction $OX'$ opposite to $OX$. The effective force in the direction $OX$ is therefore of magnitude $-P = -R\cos(180° - \theta) = -R \times (-\cos\theta) = R\cos\theta$.

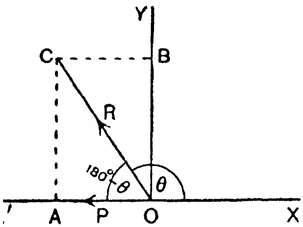



Fig. 233.

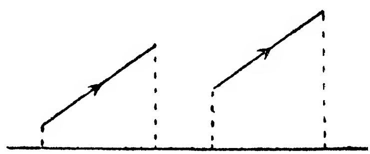


Fig. 234.

[Check.—Since $\theta$ is obtuse, $\cos\theta$ is negative, and hence the resolved part in the direction $OX$ is negative, which is correct.]

In the same way, if we take any other value of $\theta$, say in the third or fourth quadrants, we find that in every case the resolved part of $\overrightarrow{R}$ along $OX$ is $R\cos\theta$.

It may also be seen that the resolved part along $OY$ (perpendicular to $OX$) is $R\sin\theta$ in every case. Thus : *the resolved parts of a vector $R_\theta$ along $OX$, $OY$ are always $R\cos\theta$ and $R\sin\theta$.*

It is clear that the resolved parts in any given direction of two equal and parallel vectors are themselves equal (Fig. 234).

**To solve the equations $r\cos\theta = a$, $r\sin\theta = b$**

Equations of this type in which $a$ and $b$ are given, and we have to solve for $r$ and $\theta$, occur very frequently.

Since $r\cos\theta$ and $r\sin\theta$ are the resolved parts along $OX$

and $OY$ of the vector $r_\theta$, our problem is simply to find the magnitude ($r$) and direction ($\theta$) of the vector whose resolved parts along $OX$ and $OY$ are $a$ and $b$. This can be done rapidly from a rough sketch.

*Example.*—If $r\cos\theta=3$ and $r\sin\theta=4$, find $r$ and $\theta$.

The resolved parts of $r_\theta$ along $OX$, $OY$ are 3 and 4 respectively. Hence the vector is as shown in Fig. 235.

From Pythagoras' theorem,

$$r^2=3^2+4^2=9+16=25,$$

$$\therefore\ r=5.$$

(We take the positive square root for $r$ obviously.)

Also, from the figure, $\theta=\tan^{-1}\frac{4}{3}=\tan^{-1}1{\cdot}3333=53°\ 8'$.

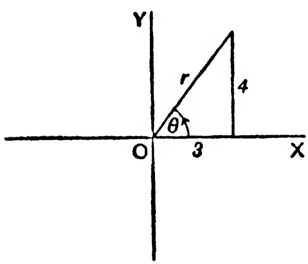



FIG. 235.

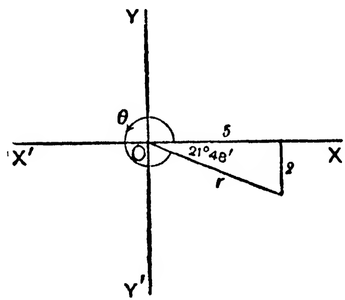



FIG. 236.

We might, of course, take $\theta$ to be $53°\ 8' \pm$ any multiple of 360°, but in every case in practice only the simplest value of $\theta$ is required.

*Example.*—Solve the equations $r\cos\theta=5$, $r\sin\theta=-2$.

The resolved part along $OY$ is negative, viz. $-2$, in this case; the vector $r_\theta$ is therefore as shown in Fig. 236,

$$r^2=5^2+2^2=25+4=29$$

$$\therefore\ r=\sqrt{29}=5{\cdot}385.$$

From the figure, the angle $\theta$ is in the fourth quadrant.

The acute angle between $OX$ and the vector

$$=\tan^{-1}\tfrac{2}{5}=\tan^{-1}0{\cdot}4=21^\circ\ 48',$$

$$\therefore\ \theta=360^\circ-21^\circ\ 48'=338^\circ\ 12'.$$

We sometimes prefer to take $\theta$ as being $-21^\circ\ 48'$. Both values for $\theta$ satisfy the equations, with the value of $r$ as found.

**Resolved part of the resultant of any number of vectors**

We shall now show that:

*The resolved part in any direction of the resultant of any number of vectors = the sum of the resolved parts, in that same direction, of the component vectors.*

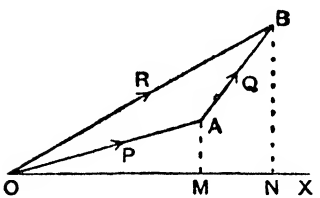


FIG. 237 (*a*).

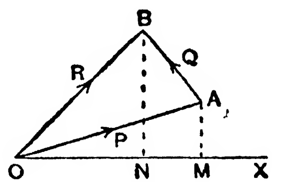


FIG. 237 (*b*).

$\vec{P}$, $\vec{Q}$ are any two vectors, and $\vec{R}$ their resultant. $OX$ is any direction; $M$, $N$ are the projections of $A$, $B$ on $OX$.

In Fig. 237 (*a*) the resolved parts of $\vec{P}$ and $\vec{Q}$ along $OX$ are $OM$, $MN$. Their sum $=OM+MN=ON=$ the resolved part of $\vec{R}$ along $OX$.

In Fig. 237 (*b*) the resolved part of $\vec{P}$ is $OM$, and the resolved part of $\vec{Q}$ along $OX$ is $-NM$ (being negative since the angle between $OX$ and $\vec{Q}$ is obtuse). Their sum $=OM-NM=ON$ $=$ the resolved part of $\vec{R}$ along $OX$.

Our statement is, in fact, true whatever the directions of $\vec{P}$ and $\vec{Q}$.

It is easily seen that the statement is equally true however many vectors there are.

### Analytical method for finding the resultant of any number of vectors

The above gives us an easy method of finding the resultant of any number of vectors *by calculation*. Take any two convenient directions $OX$, $OY$ at right angles. Resolve each force into its two components along $OX$ and $OY$. Let the algebraic sum of the components along $OX$, $OY$ be $X$ and $Y$ respectively. Then $X$ and $Y$ are the components along $OX$ and $OY$ of the resultant, which is therefore found by compounding them in the usual way. If the resultant is of magnitude $R$ and makes an angle $\theta$ with $OX$,

$$R \cos \theta = X, \qquad R \sin \theta = Y.$$

From these equations we can find $R$ and $\theta$.

We shall take the two examples which have been worked graphically on pp. 273–4.

*Example.*—Find, by calculation, the resultant of the vectors $2{\cdot}5_{30^\circ}$, $4_{70^\circ}$, $2_{125^\circ}$, $6_{212^\circ}$.

Take the line of reference as $OX$ and the line perpendicular to it (the line 90°) as $OY$.

| *Vector* | *Component along OX* | | *Component along OY* | |
|---|---|---|---|---|
| $2{\cdot}5_{30^\circ}$ | $2{\cdot}5 \cos 30^\circ = 2{\cdot}5 \times 0{\cdot}8660 =$ | $2{\cdot}1650$ | $2{\cdot}5 \sin 30^\circ = 2{\cdot}5 \times 0{\cdot}5 =$ | $1{\cdot}25$ |
| $4_{70^\circ}$ | $4 \cos 70^\circ = 4 \times 0{\cdot}3420 =$ | $1{\cdot}3680$ | $4 \sin 70^\circ = 4 \times 0{\cdot}9397 =$ | $3{\cdot}7588$ |
| $2_{125^\circ}$ | $2 \cos 125^\circ = 2 \times (-0{\cdot}5736) =$ | $-1{\cdot}1472$ | $2 \sin 125^\circ = 2 \times (0{\cdot}8192) =$ | $1{\cdot}6384$ |
| $6_{212^\circ}$ | $6 \cos 212^\circ = 6 \times (-0{\cdot}8480) =$ | $-5{\cdot}0880$ | $6 \sin 212^\circ = 6 \times (-0{\cdot}5299) =$ | $-3{\cdot}1794$ |
| | $X =$ | $-2{\cdot}7022$ | $Y =$ | $-3{\cdot}4678$ |

The resultant therefore has component $-2{\cdot}7022$ along $OX$ and $3{\cdot}4678$ along $OY$.

Drawing a rough sketch (Fig. 238) we see that:

$$R = \sqrt{(2{\cdot}7022)^2 + (3{\cdot}4678)^2}$$
$$= \sqrt{7{\cdot}302 + 12{\cdot}03} = \sqrt{19{\cdot}33} \simeq 4{\cdot}397 \simeq 4{\cdot}40.$$

Also the acute angle between $R$ and $OX'$

$$= \tan^{-1} \frac{3{\cdot}4678}{2{\cdot}7022} = 52^\circ\ 5'$$

Hence $\theta = 180^\circ - 52^\circ\ 5' = 127^\circ\ 55'.$

Thus $\vec{R} = 4{\cdot}40_{127^\circ\ 55'}.$

(This is of course more accurate than the value found graphically on p. 273.)

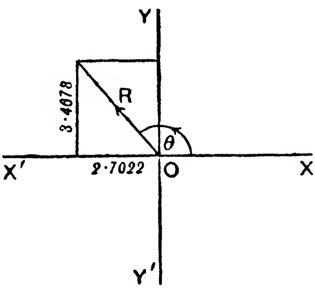



FIG. 238.

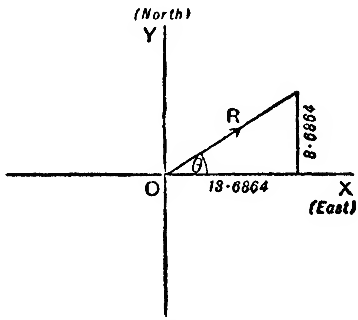



FIG. 239.

*Example.*—A body is pulled simultaneously by forces of 20 lb. wt. due N., 25 lb. wt. due E. and 16 lb. wt. in a direction S.W. Find the direction in which it moves.

Take $OX$, $OY$ in directions E. and N. respectively.

| *Vector* | *Component along OX* | *Component along OY* |
|---|---|---|
| 20 lb. wt. due N. $=20_{90^\circ}$ | $20 \cos 90^\circ = 0$ | $20 \sin 90^\circ = 20$ |
| 25 lb. wt. due E. $=25_{0^\circ}$ | $25 \cos 0^\circ = 25$ | $25 \sin 0^\circ = 0$ |
| 16 lb. wt. S.W. $=16_{225^\circ}$ | $16 \cos 225^\circ = 16 \times (-0{\cdot}7071) = -11{\cdot}3136$ | $16 \sin 225^\circ = 16 \times (-0{\cdot}7071) = -11{\cdot}3136$ |
| | $X = 13{\cdot}6864$ | $Y = 8{\cdot}6864$ |

The resultant has components 13·6864 lb. wt. due E. and 8·6864 lb. wt. due N. (shown in Fig. 239).

$\theta$ is in the first quadrant and is equal to

$$\tan^{-1}\frac{8{\cdot}6864}{13{\cdot}6864}=32^\circ\ 24'.$$

The body moves in the direction of the resultant force, i.e. in the direction N. 57° 36′ E.

*Example.*—Show that the resultant of two equal forces, each of magnitude $P$, inclined at an angle $\alpha$ to each other, is a force of magnitude $2P\cos\frac{\alpha}{2}$ in the direction bisecting the angle between them.

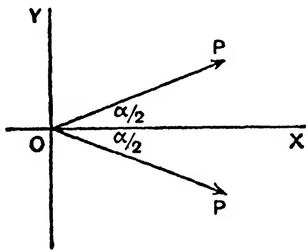



FIG. 240.

Choose the line bisecting the angle between the forces as $OX$.

The force $P_{\alpha/2}$ has components $P\cos\frac{\alpha}{2}$ along $OX$ and $P\sin\frac{\alpha}{2}$ along $OY$.

The force $P_{-\alpha/2}$ has components $P\cos\frac{\alpha}{2}$ along $OX$ and $-P\sin\frac{\alpha}{2}$ along $OY$.

Hence the resultant has components $2\,P\cos\frac{\alpha}{2}$ along $OX$ and 0 along $OY$; i.e. the resultant is a force of magnitude $2P\cos\frac{\alpha}{2}$ along the bisector of the angle.

## Exercise XXXV

[Additional easier exercises on vectors will be found in Part I (pp. 266–269).]

Find, graphically and by calculation, the magnitude and direction of the resultant of each of the following pairs of vectors, expressing its direction by the angle which it makes with the first vector in each case:

**1.** Forces of 3 lb. wt. and 10 lb. wt. inclined at 60° to each other.

**2.** Velocities of 40 ft. per sec. and 15 ft. per sec. inclined at 150°.

**3.** Forces of 5 tons wt. and 8 tons wt. at right angles.

Find, graphically and by calculation, the magnitude and direction of the resultant of the following pairs of vectors:

**4.** Forces of 10 tons wt. due N. and 7 tons wt. in the direction S.E.

**5.** Velocities of 25 m.p.h. N. 30° E. and 18 m.p.h. N. 55° W.

**6.** Express the vectors shown in Fig. 241 in the form $P_\alpha$, taking $OX$ as the direction of reference, and find their resolved parts along $OX$ and $OY$:

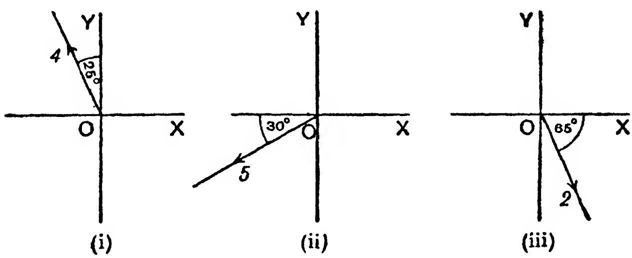



FIG. 241.

**7.** Find the resultant of the vectors $6_{20°}$ and $9_{230°}$.

**8.** A man's normal walking pace is 4 m.p.h. If he walks at his normal pace across the deck of a ship which is moving forward at 20 knots, find his actual velocity. [1 knot = a speed of 1 nautical ml. per hr. = 6080 ft. per hr. $\simeq$ 1·15 m.p.h.]

**9.** An aeroplane is climbing at an angle of 18° to the horizontal. If its horizontal speed is observed to be 135 m.p.h., what is its actual speed? How long does it take to climb 1000 ft. (vertically)?

**10.** The vector $5_{65°}$ is the resultant of two vectors, one of which is $6_{30°}$; find the other vector.

**11.** A man can row a boat at 4 m.p.h. in still water. If he wishes to row directly across a river flowing at 3 m.p.h., in what direction must he keep the boat headed?

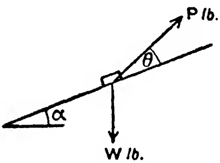


FIG. 242.

**12.** A body of weight $W$ lb. is pulled up a smooth plane, inclined at an angle $\alpha$ to the horizontal, by a force $P$ lb. wt. inclined at an angle $\theta$ to the plane. What is the resolved part of the resultant force on the body up the plane?

**13.** A force of 10 tons wt. acts in a direction N. 70° E. Find, graphically, its components in the directions E. and N.E.

**14.** Find, graphically and by calculation, the resultant of two velocities of magnitudes 20 ft. per sec. and 32 ft. per sec. in perpendicular directions.

**15.** A man walks 3 ml. due N., then 2 ml. S. 36° E. and finally 5 ml. due E. Find the distance and bearing of his final position from the starting-point.

**16.** Find, *graphically*, the resultant of the vectors $10_{60°}$, $4_{110°}$, $8_{165°}$.

**17.** Take the vectors in Question 16 in a different order and verify that the resultant is the same as that previously obtained.

**18.** Show graphically that the following three forces are in equilibrium: 7·07 tons N. 60° E., 27·32 tons N. 45° W., 26·38 tons S. 30° E.

[A number of forces are in equilibrium if their resultant is zero.]

**19.** Three forces of magnitudes 4·5 lb., 6 lb. and 9 lb. are in equilibrium. Find the angles between their directions.

Find $r$ and $\theta$ from the following equations (taking $r$ to be positive):

**20.** $r \cos \theta = 5$, $r \sin \theta = 12$. **21.** $r \cos \theta = 4$, $r \sin \theta = -3$.

22. $r\cos\theta = 2\cdot5$, $r\sin\theta = 1\cdot4$. 23. $r\cos\theta = -1\cdot6$, $r\sin\theta = 2\cdot7$.

24. Find, *by calculation*, the resultant of the vectors in Question 16.

25. A truck is pulled along a light railway by two men hauling on ropes inclined to the rails at 20° on either side. If the force along the rails necessary to move the truck is 230 lb. wt., find the pull which each man must exert.

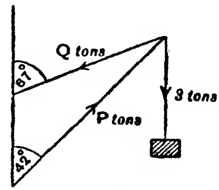



FIG. 243.

26. Fig. 243 shows a crane supporting a load of 3 tons. If the stresses in the jib and tie are $P$ tons and $Q$ tons respectively, find the values of $P$ and $Q$ *graphically*. [The three forces shown must be in equilibrium.]

27. Find $P$ and $Q$ in Question 26 *by calculation*.

28. If $P_\alpha$ is any vector, show that the resultant of the vectors $P_\alpha$, $P_{\alpha+\frac{2\pi}{3}}$, $P_{\alpha+\frac{4\pi}{3}}$ is zero.

29. From the result of Question 28 deduce, by resolving along $OX$ and $OY$ (where $OX$ is the direction of reference) that:

$$\cos\alpha + \cos\left(\alpha + \frac{2\pi}{3}\right) + \cos\left(\alpha + \frac{4\pi}{3}\right) \equiv 0$$

and
$$\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) \equiv 0.$$

[The latter identity occurs in electrical engineering in the theory of a three-phase generator.]

30. Find, *by calculation*, the resultant of the following vectors:

$$4_{15^\circ},\ 2\cdot5_{80^\circ},\ 5_{136^\circ},\ 3\cdot4_{(-70)^\circ}.$$

31. Find, *by calculation*, the magnitude and direction of the single force which is equivalent to the following forces acting at a point: 10 lb. wt. due E., 6 lb. wt. N. 24° W., 12·4 lb. wt. S. 35° W., 8·2 lb. wt. S. 57° E.

*

## CHAPTER XIII

# TRIGONOMETRIC RATIOS OF THE SUM AND DIFFERENCE OF TWO ANGLES

## RATIOS OF SMALL ANGLES

**Sine and cosine of the sum of two angles**

Let $\vec{P}$, $\vec{Q}$ be any two vectors at right angles and $\vec{R}$ their resultant. Take any line $OX$ of reference (Fig. 244) and let the angle between $OX$ and $\vec{P}$ be $\alpha$; then $\vec{P} = P_\alpha$ and $\vec{Q} = Q_{90°+\alpha}$. Let the angle between $\vec{P}$ and $\vec{R}$ be $\beta$; then $\vec{R} = R_{\alpha+\beta}$.

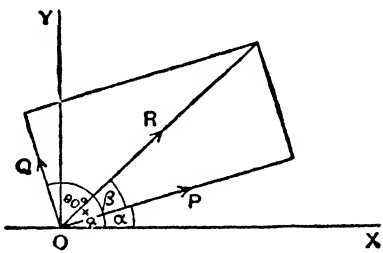



FIG. 244.

Now the resolved part of $\vec{R}$ along $OX$ = sum of resolved parts of $\vec{P}$ and $\vec{Q}$ along $OX$.

$$\therefore R \cos(\alpha + \beta) = P \cos \alpha + Q \cos (90° + \alpha)$$

$$= P \cos \alpha - Q \sin \alpha,$$

since $\cos (90° + \alpha) = -\sin \alpha$ (p. 265).

But $P = R \cos \beta$ and $Q = R \sin \beta$.

$$\therefore R \cos(\alpha + \beta) = R \cos \beta \cos \alpha - R \sin \beta \sin \alpha$$

$$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

If we take resolved parts along $OY$, instead of along $OX$, we have,

$$\begin{aligned} R \sin(\alpha+\beta) &= P \sin\alpha + Q \sin(90^\circ+\alpha) \\ &= P \sin\alpha + Q \cos\alpha \\ &= R \cos\beta \sin\alpha + R \sin\beta \cos\alpha \end{aligned}$$

$$\therefore \sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta.$$

These two formulæ are extremely important for all applications of trigonometry, and the student must commit them to memory. They are usually quoted in the forms:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

These formulæ are true whatever the angles $A$ and $B$, and whether the angles are positive or negative. The formulæ are therefore still true if we write $-B$ in place of $B$.

$$\begin{aligned} \therefore \sin(A-B) &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B + \cos A \,.\, (-\sin B) \\ &= \sin A \cos B - \cos A \sin B. \end{aligned}$$

$$\begin{aligned} \text{and } \cos(A-B) &- \cos A \cos(-B) - \sin A \sin(-B) \\ &= \cos A \cos B - \sin A \,.\, (-\sin B) \\ &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

Collecting these four formulæ we have:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B\ ;$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

where upper signs are to be taken together and lower signs together.

The student should notice the following points which will help him to remember these formulæ:

(i) the $\sin(A \pm B)$ formulæ contain "mixed products," viz. $\sin A \cos B$ and $\cos A \sin B$;

the $\cos(A \pm B)$ formulæ contain products of like ratios, viz. $\cos A \cos B$ and $\sin A \sin B$;

(ii) in the $\sin(A \pm B)$ formulæ the signs $(\pm)$ on the two sides of the identity are the same; in the $\cos(A \pm B)$ formulæ the signs are opposite.

*Example.*—Find the values of $\sin 75°$ and $\cos 75°$ without using tables.

$$\begin{aligned}
\sin 75° &= \sin(45° + 30°) \\
&= \sin 45° \cos 30° + \cos 45° \sin 30° \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{(\sqrt{3}+1)\sqrt{2}}{4} = \frac{\sqrt{6}+\sqrt{2}}{4} \\
&= \frac{2{\cdot}449 + 1{\cdot}414}{4} = \frac{3{\cdot}863}{4} \simeq 0{\cdot}966.
\end{aligned}$$

$$\begin{aligned}
\cos 75° &= \cos(45° + 30°) \\
&= \cos 45° \cos 30° - \sin 45° \sin 30° \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} \\
&= \frac{2{\cdot}449 - 1{\cdot}414}{4} = \frac{1{\cdot}035}{4} \simeq 0{\cdot}259.
\end{aligned}$$

*Example.*—Prove that $\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) = 0$ for all values of $\alpha$.

$$\begin{aligned}
&\sin\alpha + \sin\left(\alpha + \frac{2\pi}{3}\right) + \sin\left(\alpha + \frac{4\pi}{3}\right) \\
&= \sin\alpha + \left(\sin\alpha \cos\frac{2\pi}{3} + \cos\alpha \sin\frac{2\pi}{3}\right) \\
&\qquad + \left(\sin\alpha \cos\frac{4\pi}{3} + \cos\alpha \sin\frac{4\pi}{3}\right)
\end{aligned}$$

$$= \sin\alpha + (\sin\alpha\cos 120^\circ + \cos\alpha\sin 120^\circ) + (\sin\alpha\cos 240^\circ + \cos\alpha\sin 240^\circ)$$

$$= \sin\alpha + \sin\alpha \times (-\tfrac{1}{2}) + \cos\alpha \times \frac{\sqrt{3}}{2} + \sin\alpha \times (-\tfrac{1}{2}) + \cos\alpha \times \left(-\frac{\sqrt{3}}{2}\right)$$

$$= \sin\alpha - \tfrac{1}{2}\sin\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \tfrac{1}{2}\sin\alpha - \frac{\sqrt{3}}{2}\cos\alpha$$

$= 0$, for all values of $\alpha$.

[For another method of proving this identity, see Exercise XXXV, Question 29.]

*Example.*—Expand $5\sin(200t - 1{\cdot}6)$, the angle being in radians.

$$5\sin(200t - 1{\cdot}6) = 5(\sin 200t \,.\, \cos 1{\cdot}6 - \cos 200t \,.\, \sin 1{\cdot}6)$$

$$1{\cdot}6 \text{ radians} = \frac{1{\cdot}6 \times 180^\circ}{\pi} = \frac{288^\circ}{\pi} \simeq 91{\cdot}66^\circ = 91^\circ\,40' \text{ to the nearest minute.}$$

$$\therefore\ 5\sin(200t - 1{\cdot}6)$$
$$= 5(\sin 200t \cos 91^\circ\,40' - \cos 200t \sin 91^\circ\,40')$$
$$= 5\{\sin 200t \times (-0{\cdot}0291) - \cos 200t \times 0{\cdot}9996\}$$
$$= -0{\cdot}1455 \sin 200t - 4{\cdot}998 \cos 200t.$$

*Example.*—Show that $\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$.

$$\sin\left(x + \frac{\pi}{4}\right) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$$
$$= \sin x \,.\, \frac{1}{\sqrt{2}} + \cos x \,.\, \frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(\sin x + \cos x).$$

Multiply both sides by $\sqrt{2}$

$$\therefore \sin x + \cos x = \sqrt{2} \sin\left(x + \frac{\pi}{4}\right).$$

This result is often useful and is worth remembering.

**Formulæ for sin 2$A$ and cos 2$A$**

Sin $(A+B) = \sin A \cos B + \cos A \sin B$

$$\therefore \sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$$
$$= 2 \sin A \cos A$$

$$\therefore \mathbf{\sin 2A = 2 \sin A \cos A.}$$

Cos $(A+B) = \cos A \cos B - \sin A \sin B$

$$\therefore \cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A$$
$$= \cos^2 A - \sin^2 A.$$

If in this result we write $\sin^2 A = 1 - \cos^2 A$ or $\cos^2 A = 1 - \sin^2 A$ we get two other forms for $\cos 2A$; viz.:

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$$

$$\text{and } \cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

Thus

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$$
$$\mathbf{= 2\cos^2 A - 1}$$
$$\mathbf{= 1 - 2\sin^2 A.}$$

The last two formulæ express $\cos 2A$ in terms of $\cos A$ alone or $\sin A$ alone, and are very useful on that account. From them we get:

$$2\cos^2 A = 1 + \cos 2A \quad \text{and} \quad 2\sin^2 A = 1 - \cos 2A$$
$$\therefore \mathbf{\cos^2 A = \tfrac{1}{2}(1 + \cos 2A)} \quad \text{and} \quad \mathbf{\sin^2 A = \tfrac{1}{2}(1 - \cos 2A)}$$

All the formulæ printed in thick type above are important, and the student must either memorize them or be able to obtain them rapidly as we have done above.

*Example.*—Find the values of sin $22\frac{1}{2}°$ and cos $22\frac{1}{2}°$ without using tables, and compare them with the values given in the tables.

In the formula $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$ put $A = 22\frac{1}{2}°$.

$$\therefore \sin^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 - \cos 45°) = \tfrac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)$$

$$= \tfrac{1}{2}(1 - 0{\cdot}7071) = \tfrac{1}{2}(0{\cdot}2929) = 0{\cdot}14645$$

$$\therefore \sin 22\tfrac{1}{2}° = \pm\sqrt{0{\cdot}14645} = \pm 0{\cdot}3827.$$

Since $22\frac{1}{2}°$ lies in the first quadrant its sine is positive

$$\therefore \sin 22\tfrac{1}{2}° = 0{\cdot}3827.$$

From the formula for $\cos^2 A$ in terms of $\cos 2A$,

$$\cos^2 22\tfrac{1}{2}° = \tfrac{1}{2}(1 + \cos 45°) = \tfrac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \tfrac{1}{2}(1 + 0{\cdot}7071)$$

$$= \tfrac{1}{2}(1{\cdot}7071) = 0{\cdot}85355$$

$$\therefore \cos 22\tfrac{1}{2}° = \pm\sqrt{0{\cdot}85355} = \pm 0{\cdot}9239.$$

Since $22\frac{1}{2}°$ lies in the first quadrant its cosine is positive

$$\therefore \cos 22\tfrac{1}{2}° = 0{\cdot}9239.$$

On looking up sin $22\frac{1}{2}°$ and cos $22\frac{1}{2}°$ in tables we find exactly the values obtained here, so that the values we have obtained are correct to four decimal places.

*Example.*—Find expressions for $\sin 3\theta$ and $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned}
\text{Sin } 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta, \text{ using the formulæ for } \sin 2\theta \text{ and } \cos 2\theta; \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta \\
&= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta.
\end{aligned}$$

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta) \\ &= \cos 2\theta \, . \cos\theta - \sin 2\theta \sin\theta \\ &= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta \\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

The student will see that by continuing in this way we could express the sine and cosine of $4\theta$, $5\theta$, etc., in terms of $\sin\theta$ and $\cos\theta$.

## Exercise XXXVI

**1.** Verify the formulæ for $\sin(A+B)$ and $\sin(A-B)$ when $A = 70°$, $B = 40°$.

**2.** Verify the formulæ for $\cos(A+B)$ and $\cos(A-B)$ when $A = 45°$, $B = 225°$.

Use the formulæ for $\sin(A \pm B)$ and $\cos(A \pm B)$ to verify the following identities :

**3.** $\text{Sin}\,(90° + \theta) \equiv \cos\theta$.

**4.** $\text{Cos}\,(90° + \theta) \equiv -\sin\theta$.

**5.** $\text{Sin}\,(180° + \theta) \equiv -\sin\theta$.

**6.** $\text{Cos}\,(180° + \theta) \equiv -\cos\theta$.

**7.** $\text{Cos}\,(270° + \theta) \equiv \sin\theta$.

**8.** $\text{Sin}\,(180° - \theta) \equiv \sin\theta$.

**9.** If $i = 5\sin\left(300t + \frac{\pi}{6}\right)$, express $i$ in the form

$$a \sin 300t + b \cos 300t,\text{ finding the values of } a \text{ and } b.$$

Find the value of $i$ when $t = 0{\cdot}02$ from both forms and verify that the two values are equal.

**10.** Express $20\sin(100\pi t - 0{\cdot}65)$ in the form

$$a \sin 100\pi t - b \cos 100\pi t,\text{ finding the values of } a \text{ and } b.$$

**11.** Prove that $\cos\left(x - \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{3}\right) \equiv \cos x$.

**12.** If $\sin\alpha = \frac{3}{5}$, $\sin\beta = \frac{12}{13}$, and $\alpha$, $\beta$ are acute angles, find the value of $\sin(\alpha + \beta)$, without using trigonometric tables.

**13.** If $\theta$ is an acute angle and $\sin\theta = 0{\cdot}8$, find the values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, without using trigonometric tables.

Express each of the following as a single trigonometric ratio :

**14.** $\text{Sin}\,20° \cos 50° + \cos 20° \sin 50°$.

15. Cos $85^\circ \cos 25^\circ + \sin 85^\circ \sin 25^\circ$.

16. Cos $\frac{\pi}{3} \cos \frac{\pi}{6} - \sin \frac{\pi}{3} \sin \frac{\pi}{6}$.

17. Sin $10^\circ \cos 40^\circ - \cos 10^\circ \sin 40^\circ$.

18. Sin $5\theta \cos 2\theta - \cos 5\theta \sin 2\theta$.

19. Cos $\frac{\alpha}{2} \cos \frac{3\alpha}{2} - \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}$.

20. $1 - 2 \sin^2 15.^\circ$

21. Sin $\frac{x}{2} \cos \frac{x}{2}$.

22. The reading of a wattmeter is

$$W = VI\{\cos(30^\circ - \phi) + \cos(30^\circ + \phi)\}\text{watts};$$

prove that $W = \sqrt{3}\,VI \cos \phi$.

23. Prove that $\sin x - \cos x \equiv \sqrt{2} \sin\left(x - \frac{\pi}{4}\right)$.

Find the values of the following ratios, without using trigonometric tables :

24. Sin $105^\circ$.

25. Cos $15^\circ$.

26. Sin $67\frac{1}{2}^\circ$.

27. Verify that $\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$ and $\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$ when $\theta = \frac{\pi}{6}$ radians.

28. By writing $4\theta$ as $2 \times 2\theta$, prove that

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1.$$

29. The force $P$ required to pull a weight $W$ up a rough plane, of inclination $\alpha$ and coefficient of friction $\mu$, is given by $P = W(\sin \alpha + \mu \cos \alpha)$. If $\mu = \tan \lambda$, prove that $P = W\dfrac{\sin(\alpha + \lambda)}{\cos \lambda}$.

30. If the force in Question 29 is inclined at an angle $\theta$ to the plane, $P = W(\sin \alpha + \mu \cos \alpha)/(\cos \theta + \mu \sin \theta)$. Prove that $P = W \sin(\alpha + \lambda)/\cos(\theta - \lambda)$.

Prove the following identities :

31. Sin $(A + B) . \sin(A - B) \equiv \sin^2 A - \sin^2 B$.

32. (Sin $\theta + \cos \theta)^2 \equiv 1 + \sin 2\theta$.

33. Cos$^4 x - \sin^4 x \equiv \cos 2x$.

34. Cos $2\alpha - \cos 2\beta \equiv 2(\cos^2 \alpha - \cos^2 \beta) = 2(\sin^2 \beta - \sin^2 \alpha)$.

35. $\dfrac{1 - \cos \theta}{1 + \cos \theta} \equiv \tan^2 \dfrac{\theta}{2}$.

36. Cosec $x + \cot x = \cot \frac{x}{2}$.

**37.** The efficiency $e$ of a certain screw-gearing is given by $e=\dfrac{1-\mu\tan\alpha}{1+\mu\cot\alpha}$. If $\mu=\tan\lambda$, prove that $e=\tan\alpha\,.\,\cot(\alpha+\lambda)$.

Find the angles $x$ between 0° and 360° which satisfy the following equations:

**38.** $\text{Cos } 2x=\sin x$.

**39.** $\text{Sin } 2x=\sin x$.

**40.** $\text{Sin } 3x=\frac{1}{2}\sin x$.

**41.** $3\sin(x+60°)=\cos x$.

**42.** Use the result of Question 23 to find the angles between 0° and 360° which satisfy the equation $\sin x-\cos x=1$.

**43.** The distance $x$ of a piston from one end of its stroke is given by:

$$x=r(1-\cos\theta)+\frac{r^2}{2l}(1-\cos 2\theta).$$

If $r=6$ in., $l=3$ ft., find the values of $\theta$ (between 0° and 360°) for which $x=10$ in.

**44.** Solve the equation $\sin(\theta-40°)=2\cos(\theta+25°)$, giving the values of $\theta$ between 0° and 360°.

**To express $a\sin\theta+b\cos\theta$ in the form $r\sin(\theta+\alpha)$**

In the examples on p. 287 we have seen that

$$-0{\cdot}1455\sin 200t-4{\cdot}998\cos 200t=5\sin(200t-1{\cdot}6)$$

and

$$\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right).$$

We can similarly transform any expression of the type $a\sin\theta+b\cos\theta$ into the form $r\sin(\theta+\alpha)$, where $r$ and $\alpha$ have suitable values. For

$$\begin{aligned} r\sin(\theta+\alpha)&=r(\sin\theta\cos\alpha+\cos\theta\sin\alpha)\\ &=r\cos\alpha\,.\,\sin\theta+r\sin\alpha\,.\,\cos\theta. \end{aligned}$$

This is the same as $a\sin\theta+b\cos\theta$, if

$$r\cos\alpha=a \qquad \text{and} \qquad r\sin\alpha=b.$$

From these two equations we can find $r$ and $\alpha$, when $a$ and $b$ are given, as explained on p. 275.

*Example.*—Express $3\sin\theta+2\cos\theta$ in the form $r\sin(\theta+\alpha)$.

$$\begin{aligned} r\sin(\theta+\alpha)&=r(\sin\theta\cos\alpha+\cos\theta\sin\alpha)\\ &=r\cos\alpha\,.\,\sin\theta+r\sin\alpha\,.\,\cos\theta. \end{aligned}$$

This is identical with $3 \sin \theta + 2 \cos \theta$, if

$$r \cos \alpha = 3, \qquad r \sin \alpha = 2.$$

From Fig. 245, $r = \sqrt{3^2 + 2^2} = \sqrt{9+4} = \sqrt{13} \simeq 3{\cdot}606.$

$$\alpha = \tan^{-1} \tfrac{2}{3} \text{ (acute angle)}$$
$$= \tan^{-1} 0{\cdot}6667 \simeq 33° \, 42'$$

$$\therefore 3 \sin \theta + 2 \cos \theta \equiv 3{\cdot}606 \sin (\theta + 33° \, 42').$$

This shows that the result of adding the two "sine waves" $3 \sin \theta$ and $2 \cos \theta$ (the latter being a sine wave whose phase differs by a quarter of a period from that of the former) is a sine wave of amplitude 3·606, which leads the first of the two component waves by 33° 42′.

[Compare p. 247, where the curves have actually been added graphically. The resultant curve, shown in Fig. 205, has an amplitude of 3·6 (approx.) and it crosses the axis of $\theta$ from below to above when $\theta = -33°$ (approx.). Thus its equation is $y = 3{\cdot}6 \sin (\theta + 33°)$, approximately.]

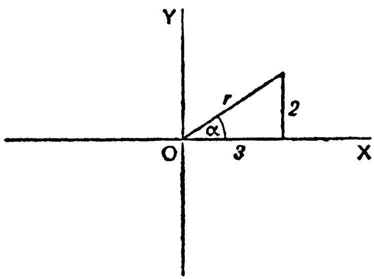



FIG. 245.

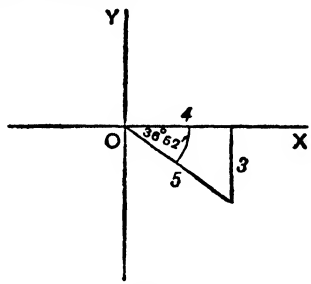



FIG. 246.

*Example.*—Convert the expression $4 \sin \omega t - 3 \cos \omega t$ into the form $r \sin (\omega t + \alpha)$.

$$r \sin (\omega t + \alpha) = r(\sin \omega t \cos \alpha + \cos \omega t \sin \alpha)$$
$$= r \cos \alpha \, . \, \sin \omega t + r \sin \alpha \, . \, \cos \omega t.$$

This is identical with $4 \sin \omega t - 3 \cos \omega t$ if

$$r \cos \alpha = 4, \qquad r \sin \alpha = -3.$$

From Fig. 246, $r = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$, and $\alpha$ is in the fourth quadrant.

The acute angle shown $= \tan^{-1} \frac{3}{4}$

$$= \tan^{-1} 0{\cdot}75 = 36° \ 52'.$$

Hence $\qquad \alpha = -36° \ 52'.$

[We could equally well write $\alpha = 360° - 36° \ 52' = 323° \ 8'$, but it is usual to take the *numerically smallest* value for $\alpha$ (i.e. the value between $-180°$ and $+180°$) in examples of this type.]

$$\therefore \ 4 \sin \omega t - 3 \cos \omega t = 5 \sin (\omega t - 36° \ 52').$$

In the general case, $a \sin \theta + b \cos \theta = r \sin (\theta + \alpha)$ where

$$r \cos \alpha = a \text{ and } r \sin \alpha = b.$$

We cannot draw the figure unless we know whether $a$ and $b$ are positive or negative; but, squaring and adding, we have

$$r^2 (\cos^2 \alpha + \sin^2 \alpha) = a^2 + b^2$$

$$\therefore \ r^2 = a^2 + b^2$$

$$\therefore \ r = \sqrt{a^2 + b^2}$$ (taking the positive value for $r$);

dividing, we have $\dfrac{r \sin \alpha}{r \cos \alpha} = \dfrac{b}{a}$.

$$\therefore \ \tan \alpha = \frac{b}{a}, \qquad \therefore \ \alpha = \tan^{-1} \frac{b}{a}.$$

The particular value of $\tan^{-1} \dfrac{b}{a}$ to be taken has to be decided by the signs of $a$ and $b$ which fix the quadrant in which $\alpha$ lies. Thus we have the general formula:

$$a \sin \theta + b \cos \theta \equiv \sqrt{a^2 + b^2} \sin \left(\theta + \tan^{-1} \frac{b}{a}\right),$$

where the value of $\tan^{-1} \dfrac{b}{a}$ is determined by the quadrant in which $\alpha$ lies.

This shows that $y = a \sin \theta + b \cos \theta$ is a sine wave of amplitude $\sqrt{a^2 + b^2}$, having a lead of $\tan^{-1} \dfrac{b}{a}$ compared with

$y = a \sin \theta$. If $\tan^{-1} \dfrac{b}{a}$ is in the third or fourth quadrant it has a lag instead of a lead.

*Example.*—Solve the equation $15 \sin \theta + 8 \cos \theta = 10$.

$15 \sin \theta + 8 \cos \theta = r \sin (\theta + \alpha)$ where $r \cos \alpha = 15$, $r \sin \alpha = 8$.

Hence, from Fig. 247, $r = \sqrt{225 + 64} = \sqrt{289} = 17$

$$\begin{aligned} \alpha &= \tan^{-1} \tfrac{8}{15} \text{ (in first quadrant)} \\ &= \tan^{-1} 0{\cdot}5333 \\ &= 28^\circ\ 4' \end{aligned}$$

$\therefore\ 15 \sin \theta + 8 \cos \theta = 17 \sin (\theta + 28^\circ\ 4')$.

The equation can therefore be written in the form

$17 \sin (\theta + 28^\circ\ 4') = 10$.

$$\begin{aligned} \therefore\ \sin (\theta + 28^\circ\ 4') &= \tfrac{10}{17} \\ &= 0{\cdot}5882 \end{aligned}$$

$$\begin{aligned} \therefore\ \theta + 28^\circ\ 4' &= 36^\circ\ 2' \pm n\ .\ 360^\circ, \\ &\text{or } 143^\circ\ 58' \pm n\ .\ 360^\circ, \end{aligned}$$

where $n$ is any integer,

$$\begin{aligned} \therefore\ \theta &= 7^\circ\ 58' \pm n\ .\ 360^\circ \\ &\text{or } 115^\circ\ 54' \pm n\ .\ 360^\circ. \end{aligned}$$

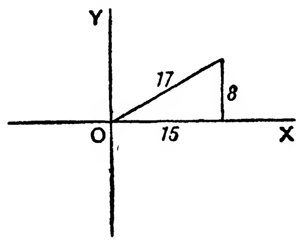



FIG. 247.

*Example.*—The displacement, $x$ in., of a slide valve is given by $x = 1{\cdot}6 + 0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t$, where $t$ is the time in sec. and the angles are in radians. Find the greatest and least values of $x$ and the times at which they occur.

$$0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t = r \sin (20\pi t + \alpha),$$

where
$$\begin{aligned} r &= \sqrt{(0{\cdot}3)^2 + (1{\cdot}1)^2} = \sqrt{0{\cdot}09 + 1{\cdot}21} \\ &= \sqrt{1{\cdot}30} = 1{\cdot}14 \end{aligned}$$

and
$$\begin{aligned} \alpha &= -\left(\text{the acute angle whose tangent is } \frac{1{\cdot}1}{0{\cdot}3}\right) \\ &= -74^\circ\ 45'. \end{aligned}$$

Since the angle $20\pi t$ is in radians we require $\alpha$ in radians.

$$\alpha = -1{\cdot}3046 \text{ (radians).}$$

$$\therefore\ 0{\cdot}3 \sin 20\pi t - 1{\cdot}1 \cos 20\pi t = 1{\cdot}14 \sin (20\pi t - 1{\cdot}3046).$$

$$\therefore\ x = 1{\cdot}6 + 1{\cdot}14 \sin (20\pi t - 1{\cdot}3046).$$

As $t$ varies, $\sin (20\pi t - 1{\cdot}3046)$ oscillates between $-1$ and $+1$, and therefore $x$ oscillates between $1{\cdot}6 - 1{\cdot}14$ and $1{\cdot}6 + 1{\cdot}14$, i.e. between $0{\cdot}46$ and $2{\cdot}74$. These are the least and greatest values of $x$.

The greatest value of $x$ occurs when $\sin (20\pi t - 1{\cdot}3046) = +1$,

i.e. when $20\pi t - 1{\cdot}3046 = \dfrac{\pi}{2} \pm n \,.\, 2\pi$, where $n$ is any integer,

$$\text{i.e. when } 20\pi t = 2{\cdot}8754 \pm n \,.\, 2\pi,$$

$$\text{i.e. when } t = \frac{2{\cdot}8754}{20\pi} \pm \tfrac{1}{10} n$$

$$= 0{\cdot}0458 \pm 0{\cdot}1 n.$$

Since $t$ denotes time, we take only positive values for $t$.

Hence $x$ attains its greatest value, viz., $2{\cdot}74$, when

$$t = 0{\cdot}0458,\ 0{\cdot}1458,\ 0{\cdot}2458, \text{ etc.}$$

Similarly, $x$ attains its least value, viz. $0{\cdot}46$, when $\sin (20\pi t - 1{\cdot}3046) = -1$,

$$\text{i.e. when } 20\pi t - 1{\cdot}3046 = -\frac{\pi}{2} \pm n 2\pi,$$

$$\text{i.e. when } 20\pi t = -0{\cdot}2662 \pm n 2n$$

$$\text{i.e. when } t = -\frac{0{\cdot}2662}{20\pi} \pm \tfrac{1}{10} n$$

$$= -0{\cdot}0042 \pm 0{\cdot}1 n,$$

$$\text{i.e. when } t = 0{\cdot}0958,\ 0{\cdot}1958,\ 0{\cdot}2958, \text{ etc.}$$

As we should expect, since the graph of $x$ against $t$ is a sine wave, these values are mid-way between the values of $t$ at which $x$ has its greatest value.

## Exercise XXXVII

Convert the following expressions into the form $r \sin(\theta+\alpha)$:

1. $7 \sin \theta + 4 \cos \theta$.
2. $1{\cdot}5 \sin \theta + 2{\cdot}8 \cos \theta$.
3. $5 \sin \theta - 12 \cos \theta$.
4. $100 \sin \theta - 82 \cos \theta$.
5. $4{\cdot}62 \cos \theta + 1{\cdot}93 \sin \theta$.

6. Express $11 \sin 300t - 25 \cos 300t$ in the form $r \sin(300t-\alpha)$, giving $\alpha$ in radians.

7. What are the greatest and least values of $a \sin \theta + b \cos \theta$ ?

8. What are the greatest and least values of $3 \sin \theta - 4 \cos \theta$, and at what values of $\theta$ (between 0° and 360°) do they occur ?

9. Express $1{\cdot}6 \sin 2\pi ft + 0{\cdot}3 \cos 2\pi ft$ in the form $r \sin(2\pi ft+\theta)$, giving $\theta$ in radians.

10. Convert $2 \cos\left(\theta - \frac{\pi}{6}\right)$ to the form $r \sin(\theta+\alpha)$.

11. Express $15 \sin 60\pi t + 8 \cos 60\pi t$ in the form $r \sin(60\pi t+\alpha)$, giving the value of $\alpha$ in radians.
Make a sketch of the graph of the function against $t$ for one complete cycle from $t=0$, indicating the period and the amplitude and the value of the function at the beginning and end of the interval.

Find the angles $\theta$ between 0° and 360° which satisfy the following equations:

12. $4{\cdot}2 \sin \theta + 5{\cdot}5 \cos \theta = 2{\cdot}7$.
13. $9 \sin \theta - 16 \cos \theta = -12$.
14. $\text{Sin } \frac{1}{2}\theta + \cos \frac{1}{2}\theta = \frac{1}{2}$.
15. $\text{Sec } \theta = 1 + 2 \tan \theta$.

16. The turning moment $T$ tons ft. on the crank shaft of an engine is given by $T = 6 + 2{\cdot}5 \sin 2\theta - 3{\cdot}8 \cos 2\theta$. Find the values of $\theta$ between 0° and 180° for which $T=3$.

17. What is the greatest value of $T$ in Question 16, and for what values of $\theta$ does it occur ?

18. Express $11{\cdot}1 \sin 3\omega t - 8{\cdot}5 \cos 3\omega t$ in the form $r \sin(3\omega t+\alpha)$, all angles being in radians; thence solve the equation $11{\cdot}1 \sin 3wt - 8{\cdot}5 \cos 3\omega t = 10$ for $t$, when $\omega = 60$.

19. For what values of $\theta$ between 0° and 360° is $3 \sin \theta + 4 \cos \theta$ positive ?

20. The force $P$ required to pull a weight $W$ up a rough plane of inclination $\alpha$ and coefficient of friction $\mu$ is given by

$$P = W(\sin \alpha + \mu \cos \alpha).$$

Find the inclination of the steepest plane up which a man exerting a force of 120 lb. wt. can pull a weight of 200 lb., the coefficient of friction being 0·4.

21. The potential difference, $v$ volts, which must be applied to send a current $I \sin \omega t$ ampères through a circuit of resistance $R$ ohms and inductance $L$ henrys is given by

$$v = I\,(R \sin \omega t + L\omega \cos \omega t).$$

Express $v$ in the form $v = IZ \sin(\omega t + \phi)$.

[$Z$ is called the *impedance* (or "apparent resistance") of the circuit.]

Find the value of $Z$ and $\phi$ if $R = 8$, $L = 0{\cdot}03$, $\omega = 100\pi$.

22. What are the greatest and least values of each of the following:

(i) $A + B \sin(\omega t + \alpha)$; (ii) $a \sin \theta + b \cos \theta + c$;

(iii) $A + B \sin(\omega t + \alpha) + C \cos(\omega t + \alpha)$?

## Approximate values of $\sin \theta$, $\cos \theta$ and $\tan \theta$ when $\theta$ is small

In this paragraph the angles are understood to be measured in *radians*; that is, $\theta$ is a number and $\sin \theta$ means $\sin(\theta \text{ radians})$, $\cos \theta$ means $\cos(\theta \text{ radians})$, and so on.

The value of the ratio $\dfrac{\sin \theta}{\theta}$ when $\theta$ is small is important for later work and so we shall draw up a table to see how this ratio behaves as $\theta$ decreases towards zero.

| $\theta$ | 1 | 0·5 | 0·2 | 0·1 |
|---|---|---|---|---|
| $\sin \theta$ .. | 0·84147 | 0·47943 | 0·19867 | 0·099831 |
| $\dfrac{\sin \theta}{\theta}$ .. | 0·84147 | 0·95886 | 0·99335 | 0·99831 |

| $\theta$ .. | 0·08 | 0·06 | 0·04 | 0·02 |
|---|---|---|---|---|
| $\sin \theta$ .. | 0·0799138 | 0·0599645 | 0·0399913 | 0·0199972 |
| $\dfrac{\sin \theta}{\theta}$ .. | 0·99892 | 0·99941 | 0·99978 | 0·99986 |

It will be noticed that as we made $\theta$ smaller and smaller we used first five-figure and then seven-figure tables, the reason being that the numerator and denominator in the fraction $\dfrac{\sin \theta}{\theta}$ were so nearly equal that greater accuracy became necessary as we proceeded.

It is seen from the table above that as $\theta$ decreases, the ratio $\frac{\sin\theta}{\theta}$ becomes more and more nearly equal to 1. The accuracy of our tables prevents us from finding the ratio when $\theta$ is less than 0·02 with a satisfactory degree of precision, but it seems reasonable to conclude that as $\theta$ approaches indefinitely closely * to 0, the ratio $\frac{\sin\theta}{\theta}$ approaches indefinitely closely to 1.

We can, in fact, *prove* by geometry that this is so. The proof is given in Part III.

We usually write the above result in the form

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\,;$$

the left-hand side is read as "the limit, as $\theta$ approaches 0, of $\frac{\sin\theta}{\theta}$."

When $\theta$ is very small, $\sin\theta$ is therefore very nearly equal to $\theta$, the radian measure of the angle.

This is very useful in finding the sines of small angles and is often used in calculations arising in surveying.

*Example.*—Find the value of sin 1° 42′ 35″.

$$1^\circ=\frac{\pi}{180}\text{ radians.}$$

$$\therefore\ 1^\circ\,42'\,35''=\left(1+\frac{42}{60}+\frac{35}{3600}\right)^\circ=\frac{6155^\circ}{3600}$$

$$=\frac{6155}{3600}\times\frac{\pi}{180}\text{ radians}=\frac{6155\times 3{\cdot}14159}{3600\times 180}\text{ radians}$$

$$=0{\cdot}02984\text{ radians.}$$

$$\therefore\ \sin 1^\circ\,42'\,35''\simeq 0{\cdot}02984.$$

* $\frac{\text{Sin}\,\theta}{\theta}$ cannot be evaluated when $\theta$ is *exactly* equal to 0, since $\frac{\sin 0}{0}=\frac{0}{0}$, which is meaningless.

This approximation is actually correct to five decimal places; by taking a more accurate value for $\pi$ we could, if required, evaluate sin 1° 42′ 35″ still more accurately.

If we tabulate the values of the ratio $\frac{\tan\theta}{\theta}$ for values of $\theta$ decreasing to 0 we obtain the following:

| $\theta$ | | 1 | 0·5 | 0·2 | 0·1 |
|---|---|---|---|---|---|
| $\tan\theta$ | .. | 1·55741 | 0·54631 | 0·20260 | 0·1003323 |
| $\frac{\tan\theta}{\theta}$ | .. | 1·55741 | 1·09262 | 1·01300 | 1·00332 |

| $\theta$ | .. | 0·08 | 0·06 | 0·04 | 0·02 |
|---|---|---|---|---|---|
| $\tan\theta$ | .. | 0·0801702 | 0·0600726 | 0·0400233 | 0·0200012 |
| $\frac{\tan\theta}{\theta}$ | .. | 1·00213 | 1·00121 | 1·00058 | 1·00006 |

It is natural to conclude that as $\theta$ approaches indefinitely closely * to 0, the ratio $\frac{\tan\theta}{\theta}$ approaches indefinitely closely to 1.

This also can be *proved* to be true (see Part III).

Thus
$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1,$$
the angle being measured in radians.

Hence when $\theta$ is very small, $\tan\theta$ is approximately equal to $\theta$.

The student will notice from the tabulated values above that when $\theta$ is very small, $\sin\theta$ is *slightly less* than $\theta$, while $\tan\theta$ is *slightly greater* than $\theta$.

When $\theta$ is very small, $\cos\theta \simeq 1$. We can obtain a better approximation however by using the approximation for $\sin\theta$.

For $\cos\theta=\cos\left(2\times\frac{\theta}{2}\right)=1-2\sin^2\frac{\theta}{2}$.

* $\frac{\tan\theta}{\theta}$ cannot be evaluated when $\theta$ is *exactly* equal to 0, since $\frac{\tan 0}{0}=\frac{0}{0}$, which is meaningless.

If $\theta$ is very small, so also is $\frac{\theta}{2}$, and therefore $\sin\frac{\theta}{2} \simeq \frac{\theta}{2}$.

$$\therefore \cos\theta \simeq 1-2\left(\frac{\theta}{2}\right)^2 = 1-2\left(\frac{\theta^2}{4}\right) = 1-\frac{\theta^2}{2}.$$

Hence we have the following approximations when $\theta$ is very small:

$$\sin\theta \simeq \theta, \quad \cos\theta \simeq 1-\frac{\theta^2}{2}, \quad \tan\theta \simeq \theta.$$

It is possible to obtain better approximations for $\sin\theta$ and $\tan\theta$. It can be proved (by methods which are too difficult to explain here) that $\sin\theta \simeq \theta - \frac{\theta^3}{6}$ and $\tan\theta \simeq \theta + \frac{\theta^3}{3}$.

*Example.*—Find the angle subtended by a circular target 5 ft. in diameter at a distance of half a mile.

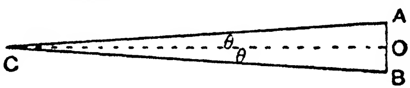



FIG. 248.

In Fig. 248, $AB$ is a diameter of the target and $O$ its centre. $CO = \frac{1}{2}$ ml. $= 2640$ ft.

If $\quad \widehat{ACO} = \theta$,

$$\tan\theta = \frac{AO}{CO} = \frac{2\frac{1}{2}}{2640} = \frac{5}{5280}.$$

Since $\theta$ is very small, $\tan\theta \simeq \theta$,

$$\therefore \theta \simeq \frac{5}{5280} \text{ radians}$$

$$= \frac{5}{5280} \times \frac{180^\circ}{\pi} = \frac{15^\circ}{88\pi}$$

$$= 0{\cdot}05426^\circ$$

$$= 3{\cdot}2556'$$

$$= 3'\ 15{\cdot}3''.$$

$$\therefore \text{Required angle} = \widehat{ACB} = 2\theta = 6'\ 30{\cdot}6''$$

$$\simeq 6'\ 31''.$$

## Exercise XXXVIII

1. Show that the approximation $\sin\theta \simeq \theta$ is correct to 3 significant figures if the angle is less than $4^\circ 20'$, and correct to 4 significant figures if it is less than $1^\circ 57'$.

2. If $x$ is small show that $\sin x' \simeq 0{\cdot}00029x$, and evaluate $\sin 24'$.

3. Find the value of $\cos 89^\circ 40'$ without tables.

4. If the distance of the Sun from the Earth is 92,000,000 miles and it subtends an angle of $32'$ at the Earth, find the diameter of the Sun.

5. The angle of depression of a ship at sea when observed from the top of a cliff 240 ft. high is $41'$. Find the distance of the ship from the foot of the cliff. (Neglect the curvature of the Earth.)

6. If an arc $AB$ of a circle subtends an angle of $20^\circ$ at the centre, find from tables the ratio length of chord $AB$ : length of arc $AB$.

7. If an arc $AB$ of a circle of radius 1 in. subtends an angle of $10^\circ$ at the centre, find the difference of the lengths arc $AB$ − chord $AB$. [Take $\pi = 3{\cdot}14159$, $\sin 5^\circ = 0{\cdot}087156$.]

8. Show that, if $\alpha$ is small, $\sin(\theta+\alpha) \simeq \sin\theta + \alpha\cos\theta$, the angles being in radians.

Deduce that $\sin\left(\frac{\pi}{4}+\alpha\right) \simeq \frac{1}{\sqrt{2}}(1+\alpha)$, and hence find the value of sin 46° to three significant figures without tables. [Take $\pi = 3{\cdot}1416$, $\sqrt{2} = 1{\cdot}4142$.]

9. Show that a better approximation to $\sin\left(\frac{\pi}{4}+\alpha\right)$ is $\frac{1}{\sqrt{2}}\left(1+\alpha-\frac{\alpha^2}{2}\right)$, and find the value of sin 46° to five significant figures by using this approximation. [Take $\pi = 3{\cdot}14159$, $\sqrt{2} = 1{\cdot}41421$.]

## Miscellaneous Exercise XXXIX

(Harder examples)

Prove the following identities:

1. $\cot\theta - \tan\theta \equiv 2\cot 2\theta$.

2. $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta} \equiv \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$.

3. $\sin^2\alpha+\sin^2\left(\alpha+\frac{2\pi}{3}\right)+\sin^2\left(\alpha+\frac{4\pi}{3}\right)\equiv\frac{3}{2}$.

4. If $\cos 2\theta=0{\cdot}4$ and $\theta$ is an acute angle, find $\sin\theta$, $\cos\theta$ and $\tan\theta$, without the aid of tables.

5. When a rod rests with its ends on two smooth planes of inclinations $\alpha$ and $\beta$, its inclination $\theta$ to the horizontal is given by the equation

$$\sin(\alpha+\beta)\,.\,\cos\theta=2\sin\alpha\,.\,\cos(\beta-\theta).$$

Prove that $$\tan\theta=\frac{\sin(\beta-\alpha)}{2\sin\alpha\sin\beta}.$$

6. The acceleration of a piston is $r\omega^2(\cos\theta\pm\frac{r}{l}\cos 2\theta)$, the $+$ and $-$ signs corresponding to out-stroke and in-stroke respectively. If $\frac{r}{l}=\frac{1}{5}$, find the values of the crank angle $\theta$ for which the acceleration of the piston is zero.

7. The equation $\dfrac{\sin(\beta+\theta)}{\sin(\beta-\theta)}=\dfrac{\cos(30^\circ-\phi)}{\cos(30^\circ+\phi)}$ occurs in the design of a power-factor meter for alternating currents. Show that $\tan\theta=\dfrac{1}{\sqrt{3}}\tan\beta\tan\phi$.

8. Express $3\cos\omega t-4\cos(\omega t-60^\circ)$ in the form

$$a\sin\omega t+b\cos\omega t,$$

finding the values of $a$ and $b$, and thence convert the expression into the form $c\sin(\omega t+\alpha)$.

9. Find numbers $a$ and $b$ which make

$a\sin(\theta-30^\circ)+b\sin(\theta+60^\circ)$ identically equal to $2\sin\theta$.

10. If $\sin\theta+\cos\phi=p$ and $\cos\theta-\sin\phi=q$, show that $\sin(\theta-\phi)=\frac{1}{2}(p^2+q^2)-1$.

11. Find all the values of $x$ between 0 and 10 which satisfy the equation $6\sin\dfrac{\pi x}{4}+8\cos\dfrac{\pi x}{4}=5$.

12. Prove that:

$$\lim_{h\to 0}\left\{\frac{\sin(x+h)-\sin(x-h)}{2h}\right\}=\cos x,$$

the angles being measured in radians.

# CHAPTER XIV

## TRIANGLE FORMULÆ

## SOLUTION OF TRIANGLES

### Relations between the sides and angles of a triangle

The three sides and the three angles of a triangle are often called its six "parts" or "elements."

In order to construct a triangle we do not need to know all the six parts; for example, it will be sufficient if we know the lengths of the three sides. The three angles are therefore determined by the lengths of the sides, and hence it must be possible to find a formula expressing the angles of any triangle in terms of the sides. There are also other relations, the most important of which are given below.

If $ABC$ is any triangle, the lengths of the sides $BC$, $CA$, $AB$ are usually denoted by $a$, $b$, $c$, respectively. [NOTE.—$a$ is the side opposite the angle $A$, $b$ opposite $B$ and $c$ opposite $C$.]

### The Sine Rule

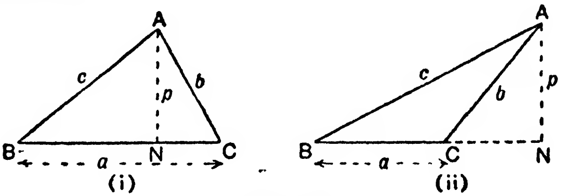


FIG. 249.

In Fig. (i) the triangle is acute-angled, in Fig. (ii) it is obtuse-angled. In each case $AN$ is perpendicular to $BC$ (or $BC$ produced).

In Fig. (i):

$$\frac{p}{c} = \sin B$$

In Fig. (ii):

$$\frac{p}{c} = \sin B$$

$\therefore p = c \sin B$ $\qquad$ $\therefore p = c \sin B$

$\frac{p}{b} = \sin C$ $\qquad$ $\frac{p}{b} = \sin(180° - C) = \sin C$

$\therefore p = b \sin C$ $\qquad$ $\therefore p = b \sin C.$

Hence in each case,

$$b \sin C = c \sin B$$

$$\therefore \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Similarly, by drawing the perpendicular from $C$ to $AB$, it can be shown that $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

We may express the *sine rule* in words thus :

*The sides of a triangle are proportional to the sines of the opposite angles.*

*Example.*—If, in a triangle $ABC$, $BC = 4{\cdot}3$ in., $\angle A = 65°$, $\angle B = 36°$, find the length of the side $AC$.

Here we are given that $a = 4{\cdot}3$ in. and we want to find $b$.

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore b = \frac{a \sin B}{\sin A}$$

$$= \frac{4{\cdot}3 \sin 36°}{\sin 65°}$$

$$= 2{\cdot}789$$

$$\therefore AC \simeq 2{\cdot}79 \text{ in.}$$

| *No.* | *Log.* |
|---|---|
| 4·3 | 0·6335 |
| sin 36° | $\bar{1}$·7692 |
| | 0·4027 |
| sin 65° | $\bar{1}$·9573 |
| 2·789 | 0·4454 |

*Example.*—$P$, $Q$ are two ends of a surveyor's base-line, 1200 yd. long, and a landmark $R$ is observed from $P$ and $Q$.

The angles $QPR$ and $PQR$ are found to be 28° and 105° 20′ respectively. What is the distance of $R$ from $P$, and what is its shortest distance from the line $PQ$ ?

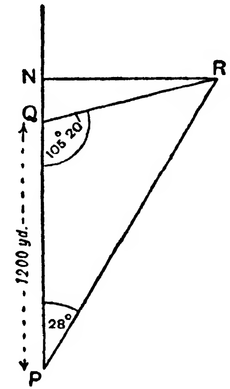



FIG. 250.

In the triangle $PQR$,

$$\begin{aligned} P\widehat{R}Q &= 180° - (28° + 105°\ 20') \\ &= 180° - 133°\ 20' \\ &= 46°\ 40' \end{aligned}$$

But $$\frac{PR}{\sin P\widehat{Q}R} = \frac{PQ}{\sin P\widehat{R}Q}$$

$$\therefore \frac{PR}{\sin 105°\ 20'} = \frac{1200}{\sin 46'\ 40'}$$

$$\therefore PR = \frac{1200 \sin 105°\ 20'}{\sin 46°\ 40'}$$

$$= \frac{1200 \sin 74°\ 40'}{\sin 46°\ 40'}$$

$$\simeq \underline{\underline{1591 \text{ yd.}}}$$

| *No.* | *Log.* |
|---|---|
| 1200 | 3·0792 |
| sin 74° 40′ | $\bar{1}$·9842 |
| | 3·0634 |
| sin 46° 40′ | $\bar{1}$·8618 |
| 1591 | 3·2016 |

Draw $RN$ perpendicular to $PQ$ produced.

In $\triangle PRN$,

| | No. | Log. |
|---|---|---|
| $\dfrac{RN}{PR}=\sin 28°$ | 1591 | 3·2016 |
| $\therefore\ RN=PR\sin 28°=1591\sin 28°$ | sin 28° | $\bar{1}$·6716 |
| $\simeq 746{\cdot}7$ yd. | 746·7 | 2·8732 |

### The Cosine Rule

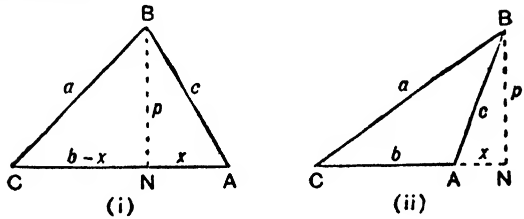



FIG. 251.

In Fig. (i), the angle $A$ is acute; in Fig. (ii) the angle $A$ is obtuse.

Draw $BN$ perpendicular to $CA$, or $CA$ produced.

Let $AN=x$. Then,

in Fig. (i) $CN=b-x$.

From Pythagoras' theorem,

$$a^2=(b-x)^2+p^2$$
$$c^2=x^2+p^2$$
$$\therefore\ a^2-c^2=(b-x)^2-x^2$$
$$=b^2-2bx$$

But $\dfrac{x}{c}=\cos A$

$$\therefore\ x=c\cos A$$
$$\therefore\ a^2-c^2=b^2-2bc\cos A$$

in Fig. (ii) $CN=b+x$.

From Pythagoras' theorem,

$$a^2=(b+x)^2+p^2$$
$$c^2=x^2+p^2$$
$$\therefore\ a^2-c^2=(b+x)^2-x^2$$
$$=b^2+2bx.$$

But $x=c\cos(180°-A)$

$$\therefore\ x=-c\cos A$$
$$\therefore\ a^2-c^2=b^2-2bc\cos A.$$

Hence in each case,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

In a similar manner we can prove that,

$$b^2 = c^2 + a^2 - 2ca \cos B$$

and $$c^2 = a^2 + b^2 - 2ab \cos C.$$

From the formula $a^2 = b^2 + c^2 - 2bc \cos A$, we have,

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Similarly, $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ and $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

[Note that the angle $A$ is contained between the sides $b$ and $c$, the angle $B$ between the sides $c$ and $a$, the angle $C$ between the sides $a$ and $b$; thus the denominator in the formula for $\cos A$ is twice the product of the sides containing the angle.]

The cosine formula enables us to find the angles of a triangle when its sides are given.

*Example.*—Find the largest angle of the triangle whose sides are 8 cm., 11 cm., 10 cm.

The largest angle is the one opposite the longest side, that is the angle opposite the side of length 11 cm.

Let $a = 8$, $b = 11$, $c = 10$.

The required angle is $B$.

$$\begin{aligned} \text{Cos } B &= \frac{c^2 + a^2 - b^2}{2ca} \\ &= \frac{10^2 + 8^2 - 11^2}{2 \times 10 \times 8} = \frac{100 + 64 - 121}{160} = \frac{43}{160} \\ &= 0{\cdot}2688 \end{aligned}$$

$$\therefore B = 74^\circ\ 24'.$$

*Example.*—In the crane represented in Fig. 252, the upright $AB$ is 10 ft., the tie-rod is 22 ft. long, and the angle $BAC$ is 115°. Find the length of the jib $BC$.

Here $c=10$, $b=22$, $\widehat{A}=115°$.

$\therefore\ a^2=b^2+c^2-2bc\cos A$
$=484+100-440\cos 115°$
$=584+440\cos 65°$, since $\cos 115°=-\cos 65°$.
$=584+(440\times 0{\cdot}4226)$
$=584+185{\cdot}944$
$=769{\cdot}944$

$\therefore\ a \simeq 27{\cdot}75$

$\therefore\ BC \simeq 27{\cdot}75$ ft.

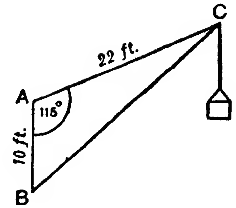



Fig. 252.

*Example.*—The crank of a steam-engine is 1 ft. long and the connecting-rod is 4 ft. 6 in. long. If the crank makes an angle of 55° with the line of stroke of the piston, find (i) the angle between the connecting-rod and the line of stroke, (ii) the distance of the crosshead from the end of its stroke.

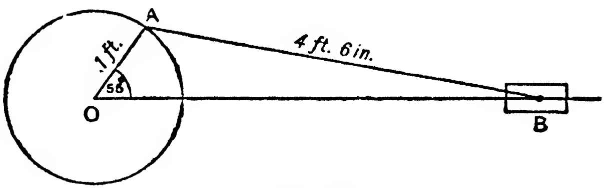



Fig. 253.

We can find $A\widehat{B}O$ from the sine rule:

$$\frac{\sin A\widehat{B}O}{AO}=\frac{\sin A\widehat{O}B}{AB}$$

$$\therefore\ \frac{\sin A\widehat{B}O}{1}=\frac{\sin 55°}{4{\cdot}5}$$

$$\therefore\ \sin A\widehat{B}O=\frac{0{\cdot}8192}{4{\cdot}5}$$
$$=0{\cdot}1820$$

$\therefore\ \underline{A\widehat{B}O=10°\,29'}$, since, from the figure, it must be acute.

Hence $O\widehat{A}B=180°-(A\widehat{O}B+A\widehat{B}O)=180°-65°\,29'=114°\,31'$.

We can now find $OB$ either from the cosine rule or from the sine rule.

Using the cosine rule,

$$OB^2 = OA^2 + AB^2 - 2OA \,.\, AB \cos O\widehat{A}B$$
$$= 1 + (4{\cdot}5)^2 - 9 \cos 114° \, 31'$$
$$= 1 + 20{\cdot}25 + 9 \cos 65° \, 29'$$
$$= 1 + 20{\cdot}25 + (9 \times 0{\cdot}4150)$$
$$= 1 + 20{\cdot}25 + 3{\cdot}7350$$
$$= 24{\cdot}985$$
$$\therefore OB \simeq 5{\cdot}00 \text{ ft.}$$

When $B$ is at the end of its stroke, $OB = OA + AB = 5$ ft. 6 in.

Hence in the given position, $B$ is 6 in. from the end of its stroke.

**Resultant of two vectors**

If $AB$, $AC$ are two vectors of magnitudes $P$, $Q$, inclined to each other at an angle $\alpha$, their resultant $AD$ is obtained by the parallelogram law. Suppose its magnitude is $R$ and that its direction makes an angle $\theta$ with that of $AB$.

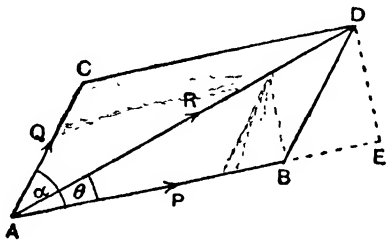



FIG. 254.

In $\triangle ABD$, $AD^2 = AB^2 + BD^2 - 2AB \,.\, BD \cos A\widehat{B}D$.

But $BD = AC = Q$ and $A\widehat{B}D = 180° - B\widehat{A}C = 180° - \alpha$.

$$\therefore R^2 = P^2 + Q^2 - 2PQ \cos (180° - \alpha)$$
$$= P^2 + Q^2 + 2PQ \cos \alpha$$
$$\therefore R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}.$$

Draw the perpendicular, $DE$, from $D$ to $AB$ produced.

$$DE = BD \sin E\widehat{B}D = Q \sin \alpha$$

$$BE = BD \cos E\widehat{B}D = Q \cos \alpha$$

$$\therefore AE = AB + BE = P + Q \cos \alpha$$

$$\therefore \tan \theta = \frac{DE}{AE} = \frac{Q \sin \alpha}{P + Q \cos \alpha}.$$

Fig. 254 has been drawn for the case in which $\alpha$ is an acute angle. The student should verify that the formulæ for $R$ and $\tan \theta$ are still true if $\alpha$ is an obtuse angle.

### Area of a triangle

The area of a triangle is often denoted by $\triangle$ (Greek capital "delta").

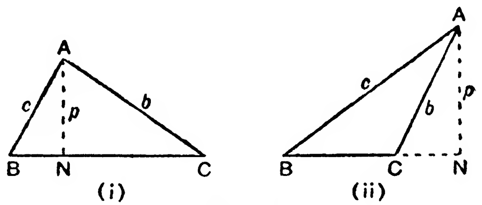



FIG. 255.

In each of Figs. (i) and (ii), $\triangle = \frac{1}{2}BC \cdot AN = \frac{1}{2}a \cdot p$.

In Fig. (i), $\frac{p}{b} = \sin C$.

In Fig. (ii), $\frac{p}{b} = \sin A\widehat{C}N = \sin (180° - C) = \sin C$.

Hence in each case, $p = b \sin C$.

$$\therefore \triangle = \tfrac{1}{2}ab \sin C$$

Similarly it can be proved that $\triangle = \frac{1}{2}bc \sin A$ and $\triangle = \frac{1}{2}ca \sin B$.

**Thus, the area of a triangle = $\frac{1}{2}$ product of any two sides × sine of angle between them.**

### Area of a regular polygon

A regular polygon is one whose sides are all equal and whose angles are all equal.

A circle can be drawn to circumscribe a regular polygon, i.e. to pass through all its vertices. [Fig. 256 shows a regular

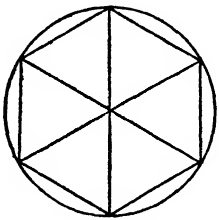


FIG. 256.

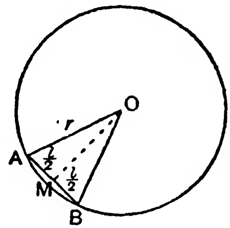



FIG. 257.

hexagon and its circumscribing circle.] By joining each of the vertices to the centre of the circle, the polygon is divided up into a number of equal triangles, and the area of the polygon is the sum of the areas of those triangles.

Suppose the polygon has $n$ sides, each of length $l$. Let $O$ be the centre of its circumscribing circle and $OAB$ (Fig. 257) one of the $n$ triangles into which the polygon is divided.

Let $M$ be the mid-point of $AB$; then $OM$ is perpendicular to $AB$.

$$A\hat{O}B = \frac{2\pi}{n} \text{ radians} = \frac{360^\circ}{n}.$$

$$\therefore A\hat{O}M = \frac{\pi}{n} \text{ radians} = \frac{180^\circ}{n}.$$

$$AM = \frac{l}{2}. \quad \therefore OM = \frac{l}{2} \cot \frac{\pi}{n}.$$

$$\therefore \text{Area } \triangle AOB = \tfrac{1}{2} AB \times OM = \frac{l}{2} \times \frac{l}{2} \cot \frac{\pi}{n} = \tfrac{1}{4} l^2 \cot \frac{\pi}{n}.$$

$$\therefore \text{Area of polygon} = n \times \text{area } \triangle AOB = \underline{\tfrac{1}{4} n l^2 \cot \frac{\pi}{n}}.$$

If the radius of the circumscribing circle is $r$, we can express the area as follows:

$$\text{Area of } \triangle AOB = \tfrac{1}{2}OA \,.\, OB \sin A\hat{O}B = \tfrac{1}{2}r^2 \sin \frac{2\pi}{n}.$$

$$\therefore \text{ Area of polygon} = n \times \text{area } \triangle AOB = \underline{\tfrac{1}{2}nr^2 \sin \frac{2\pi}{n}}.$$

**Area of a segment of a circle**

By a *segment* of a circle is meant the part cut off by a chord. Any chord such as $AB$ in Fig. 258 divides a circle into two segments, a "major segment" $ADB$ and a "minor segment" $ACB$. In Fig. 258 the minor segment cut off by the chord $AB$ is shown shaded.

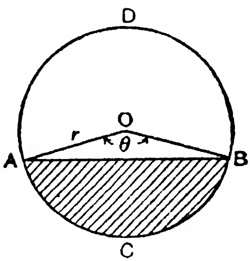



FIG. 258.

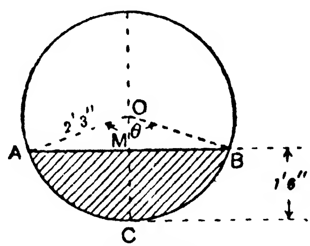



FIG. 259.

Let the arc of the segment subtend an angle $\theta$ radians at the centre of the circle; let the radius of the circle be $r$.

Area of segment $ACB$

$$\begin{aligned} &= \text{area of sector } OACB - \text{area of } \triangle OAB \\ &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}OA \,.\, OB \sin\theta \\ &= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta \\ &= \tfrac{1}{2}r^2(\theta - \sin\theta). \end{aligned}$$

*Exercise.* —Verify that this formula is true also for a major segment, where $\theta$ is the angle (in radians) subtended by the arc of the major segment. [In Fig. 258 the angle subtended by the major segment $ADB$ is the reflex angle $AOB$.]

*Example.*—A cylindrical boiler 10 ft. long and 4 ft. 6 in. in diameter, with its axis horizontal, contains water to a depth of 1 ft. 6 in. Find the volume of the water.

A cross-section of the boiler is shown in Fig. 259.

$$OM = OC - MC$$
$$= 2'\,3'' - 1'\,6'' = 9''.$$

If $\qquad \widehat{AOB} = \theta,\ \widehat{AOM} = \dfrac{\theta}{2}.$

$$\therefore \cos\frac{\theta}{2} = \frac{OM}{OA} = \frac{\frac{3}{4}}{2\frac{1}{4}} = \frac{3}{9} = \frac{1}{3} = 0{\cdot}3333$$

$$\therefore \frac{\theta}{2} = 70^\circ\ 32' = 1{\cdot}2310 \text{ radians}$$

$$\therefore \theta = 2{\cdot}462 \text{ radians}$$

$\therefore$ Area of sector $OACB = \frac{1}{2}r^2\theta = \frac{1}{2} \times (2{\cdot}25)^2 \times 2{\cdot}462$
$= 6{\cdot}233$ sq. ft.

Area of $\triangle OAB$

$= \frac{1}{2} . OA . OB \sin\theta$

$= \frac{1}{2} \times (2{\cdot}25)^2 \times \sin 141^\circ\ 4'$

$= \frac{1}{2} \times (2{\cdot}25)^2 \times \sin 38^\circ\ 56'$

$= 1{\cdot}591$ sq. ft.

| No. | Log. |
|---|---|
| 2·25 | 0·3522 |
| $(2{\cdot}25)^2$ | 0·7044 |
| sin 38° 56′ | $\bar{1}{\cdot}7982$ |
| | 0·5026 |
| 2 | 0·3010 |
| 1·591 | 0·2016 |

$\therefore$ Area of cross-section of water (shaded area)

$= 6{\cdot}233 - 1{\cdot}591 = 4{\cdot}642$ sq. ft.

$\therefore$ Volume of water $= 10 \times 4{\cdot}642$

$= 46{\cdot}42$ cu. ft.

**Formula for the area of a triangle in terms of the sides**

We have seen that $\triangle = \frac{1}{2}ab \sin C$.

But $\sin^2 C = 1 - \cos^2 C$

$$= (1 + \cos C)(1 - \cos C)$$

$$=\left(1+\frac{a^2+b^2-c^2}{2ab}\right)\left(1-\frac{a^2+b^2-c^2}{2ab}\right)$$

$$=\left(\frac{a^2+b^2+2ab-c^2}{2ab}\right)\left(\frac{c^2-a^2-b^2+2ab}{2ab}\right)$$

$$=\frac{\{(a+b)^2-c^2\}\{c^2-(a-b)^2\}}{4a^2b^2}$$

$$=\frac{(a+b+c)(a+b-c)(c+a-b)(c-a+b)}{4a^2b^2}$$

It is usual to write $a+b+c\equiv 2s$, so that $s$ is the semi-perimeter of the triangle.

Then $a+b-c=2s-2c=2(s-c)$, and similarly

$$a-b+c=2(s-b) \quad \text{and} \quad b+c-a=2(s-a).$$

Hence $$\sin^2 C=\frac{16s(s-a)(s-b)(s-c)}{4a^2b^2}$$

$$\therefore \sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)},$$

and, substituting in the formula $\triangle=\frac{1}{2}ab\sin C$, we have

$$\triangle=\sqrt{s(s-a)(s-b)(s-c)}$$

*Example.*—The field $ABCD$ represented in Fig. 260 has the following dimensions: $AB=325$ yd., $BC=100$ yd., $CD=240$ yd., $DA=210$ yd.; $\angle A=70°$. Find its area.

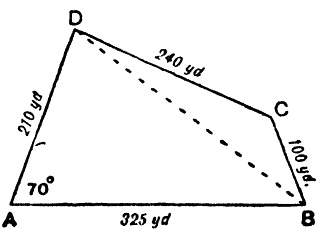



FIG. 260.

Divide the area up into two triangles by joining $BD$.

Area $\triangle ABD$

$= \frac{1}{2} \times 210 \times 325 \times \sin 70°$

$= 105 \times 325 \times \sin 70°$

$\simeq 32{,}070$ sq. yd.

| *No.* | *Log.* |
|---|---|
| 105 | 2·0212 |
| 325 | 2·5119 |
| sin 70° | $\bar{1}$·9730 |
| 32070 | 4·5061 |

To find the area of $\triangle BCD$ we need to know either the angle $BCD$ or the side $BD$.

Applying the cosine formula to $\triangle ABD$,

$BD^2 = 210^2 + 325^2 - 2 \times 210 \times 325 \cos 70°$

$= 44100 + 105625 - 46690$

$= 103035$

$\therefore BD \simeq 321$ yd.

| | |
|---|---|
| 420 | 2·6232 |
| 325 | 2·5119 |
| cos 70° | $\bar{1}$·5341 |
| 46690 | 4·6692 |

Denoting the sides of $\triangle BCD$ by $a, b, c$,

$a = 240,\ b = 321,\ c = 100.$

$\therefore s = \frac{1}{2}(a + b + c) = (240 + 321 + 100)$

$= \frac{1}{2} \times 661 = 330·5.$

$\therefore s - a = 90·5,\ s - b = 9·5,\ s - c = 230·5$

$\therefore$ Area $\triangle BCD$

$= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{330·5 \times 90·5 \times 9·5 \times 230·5}$

$\simeq 8{,}091$ sq. yd.

$\therefore$ Area of field

$\simeq 32{,}070 + 8{,}091$

$= 40{,}161$ sq. yd. $\simeq 8·3$ acres.

| | |
|---|---|
| 330·5 | 2·5191 |
| 90·5 | 1·9566 |
| 9·5 | 0·9777 |
| 230·5 | 2·3626 |
| | 7·8160 |
| 8091 | 3·9080 |

**Area of a quadrilateral**

Since the lengths of the sides of a quadrilateral do not fix its shape it is not possible to express the area of a quadrilateral in terms of its sides.

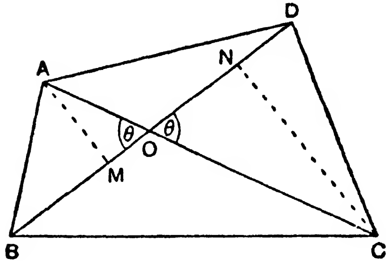



FIG. 261.

The following formula is however sometimes useful.

In Fig. 261, $AM$ and $CN$ are drawn perpendicular to $BD$.

Let the angle between the diagonals be $\theta$.

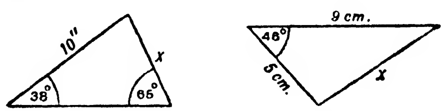
Area of quadrilateral $ABCD$

$$\begin{aligned}&=\text{area } \triangle ABD + \text{area } \triangle BCD\\&=\tfrac{1}{2}BD\,.\,AM+\tfrac{1}{2}BD\,.\,CN\\&=\tfrac{1}{2}BD\,.\,AO\sin\theta+\tfrac{1}{2}BD\,.\,CO\sin\theta\\&=\tfrac{1}{2}BD(AO+CO)\sin\theta\\&=\tfrac{1}{2}BD\,.\,AC\sin\theta.\end{aligned}$$

That is, in words, **area of a quadrilateral $=\frac{1}{2}\times$ product of diagonals $\times$ sine of angle between them.**

## Exercise XL

**1.** Find the side $x$ in the triangles shown :—




(i) (ii)

FIG. 262.

**2.** Two angles of a triangle are 46° and 73°, and the side opposite the smaller angle measures 8 cm. What is the length of the side opposite the larger angle?

3. In a $\triangle PQR$, $\angle P = 34° 30'$, $\angle Q = 68°$ and $PQ = 5$ in. Find the length of $QR$.

4. The sides of a triangle measure 13 ft., 26 ft., 19 ft. Find the smallest angle of the triangle.

5. Two sides of a triangle measure 18 in. and 23 in. and contain an angle 62° 10′ between them. What is the length of the third side of the triangle?

6. The sides of a triangle are in the ratio 3 : 5 : 7. Find the greatest angle.

7. Two angles of a triangle are 39° and 114°, and its shortest side is 7·5 cm. Find its longest side.

8. Find the area of the triangle in Question 4.

9. Find the area of the triangle in Question 3.

10. In a $\triangle DEF$, $DE = 8{\cdot}2$ in., $\angle D = 61°$, $\angle E = 43°$. Find the length of $DF$ and the area of the triangle.

11 Two landmarks $A$, $B$ are on opposite sides of a mountain, but both are visible from a point $C$. It is found that $AC = 2$ ml. 320 yd., $CB = 3$ ml. 850 yd. and $\angle ACB = 57° 40'$. What is the distance from $A$ to $B$ as the crow flies?

12. A trench is 6 ft. 6 in. deep and 3 ft. 6 in. wide at the bottom, and the sides slope at 80° to the horizontal. Find the length of the shortest plank which will reach from the bottom of one side to the top of the opposite side.

13. The dimensions of a derrick-crane are as shown in Fig. 263. Find the height of $A$ above $BC$.

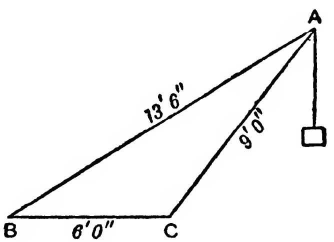



FIG. 263.

14. A motor-car door is prevented from swinging open to its fullest extent by a strap 10 in. long. The ends of the strap are fixed to points 6 in. and $4\frac{1}{2}$ in. from the line of hinges. Find the greatest angle through which the door can open.

**15.** A wireless station $P$ is 70 miles due North of another station $Q$. A ship finds that it is S. 28° 12′ W. from $P$ and N. 67° 35′ W. from $Q$. Find its distance from $P$.

**16.** The cantilever structure in Fig. 264 has the dimensions shown. Find the lengths of the members $BC$, $AC$, $CD$.

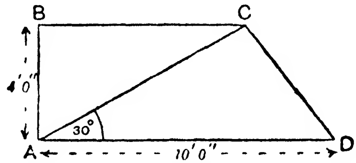



FIG. 264.

**17.** Two forces of 4 lb. wt. and 6 lb. wt. act in directions inclined at 50° to each other. What is the magnitude of the resultant force ?

**18.** An aeroplane, whose maximum speed in the absence of any wind is 240 m.p.h., flies at full throttle heading in a direction N.E. The pilot finds that his actual direction is N. 29° E. and his actual speed 219 m.p.h. Find the velocity and direction of the wind.

**19.** Find the magnitude and direction of the resultant of two forces of 2 tons wt. and 5 tons wt. in directions inclined at 110° to each other.

**20.** Two sides of an acute-angled triangle measure 8·2 in. and 9·9 in., and its area is 31·5 sq. in. Find the length of the third side. [Hint.—First find the angle between the given sides.]

**21.** Find the area of a regular octagon of side 4 cm.

**22.** A regular 7-sided polygon is inscribed in a circle of radius 3 in. Find its perimeter and its area.

**23.** A hexagonal nut fits into a $\frac{3}{4}$-in. spanner and screws on to a $\frac{1}{2}$-in. bolt. Find the area of its section.

**24.** A regular pentagon has an area of 25 sq. cm. Find the length of its side.

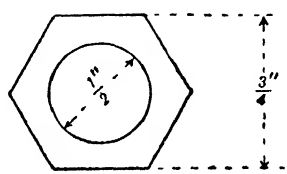



FIG. 265.

**25.** A gutter whose cross-section is an arc of a circle is 2 in. deep and 6 in. wide across the top (Fig. 266). Find the area of its cross-section.

**26.** A straight tunnel for an underground railway is to be 600 yd. long and its cross-section is to have the dimensions shown in Fig. 267. Find the volume of earth to be excavated.

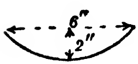



FIG. 266.

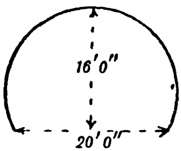



FIG. 267.

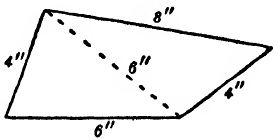



FIG. 268.

**27.** Find the area of the metal plate in Fig. 268, from the dimensions given, using the formula on p. 315.

**28.** Draw Fig. 268 to scale. Measure the other diagonal and the angle between the diagonals, and calculate its area from the formula on p. 317. Compare this with the value obtained in Question 27.

## Solution of triangles

A triangle can be constructed if certain of its parts are given, such as its three sides, or two sides and the included angle, or one side and two angles (see Part I, Ch. XI). When three such parts are given the remaining three parts can be found either graphically, by drawing the triangle, or more accurately by calculation, using the sine and cosine rules. We usually speak of this as "solving" the triangle.

The cases in which it is possible to solve a triangle are set out below.

### I. Given the three sides

The angles can be calculated from the cosine rule.

It is only necessary to calculate two of the angles in this way, since the third angle can then be found from the fact that the sum of the three angles is equal to 180°. The student

**may, however, prefer to calculate all three angles and to use the fact about the angle-sum being equal to 180° as a check.**

*Example.*—Solve the following triangle: $a=6{\cdot}3$ in., $b=2{\cdot}9$ in., $c=8{\cdot}2$ in.

$$\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{8{\cdot}41+67{\cdot}24-39{\cdot}69}{2\times 2{\cdot}9\times 8{\cdot}2}$$

$$=\frac{35{\cdot}96}{47{\cdot}56}=0{\cdot}7561$$

$$\therefore \underline{A=40^\circ\ 53'}$$

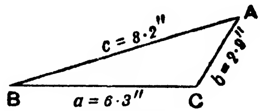


FIG. 269.

$$\cos B=\frac{c^2+a^2-b^2}{2ca}=\frac{67{\cdot}24+39{\cdot}69-8{\cdot}41}{2\times 8{\cdot}2\times 6{\cdot}3}$$

$$=\frac{98{\cdot}52}{103{\cdot}32}=0{\cdot}9535$$

$$\therefore \underline{B=17^\circ\ 32'}$$

$$C=180^\circ-(A+B)=180^\circ-58^\circ\ 25'$$

$$=\underline{121^\circ\ 35'}.$$

[*or*
$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{39{\cdot}69+8{\cdot}41-67{\cdot}24}{2\times 6{\cdot}3\times 2{\cdot}9}$$

$$=-\frac{19{\cdot}14}{36{\cdot}54}=-0{\cdot}5237.$$

Since $\cos C$ is negative, the angle $C$ is obtuse.

$$\therefore C=180^\circ-58^\circ\ 25'=121^\circ\ 35'.$$

*Check :* $A+B+C=40^\circ\ 53'+17^\circ\ 32'+121^\circ\ 35'=180^\circ\ 0'$.]

As an alternative method, having found one angle by the cosine rule, we may then find the second angle by using the sine rule, which is more suited for logarithmic calculations.

Care must be exercised however in finding angles from the sine rule, since an angle and its supplement have the same sine; so that, if we are given the sine of an angle, there are

two angles, each less than 180°, having that sine, one acute and the other obtuse. We have then to decide in some way which is the correct angle to take. We can avoid any ambiguity by calculating the two smaller angles of the triangle, as we know that they must both be acute, since a triangle cannot have more than one obtuse angle (if any).

Thus, in the previous example, the smallest angle, being opposite the smallest side, is $B$.

From the cosine formula, we find (as above) that $B = \underline{17^\circ\ 32'}$. The next angle in order of magnitude is $A$ (opposite the second shortest side).

$$\frac{\sin A}{\sin B} = \frac{a}{b}$$

$$\therefore \sin A = \frac{a \sin B}{b}$$

$$= \frac{6{\cdot}3 \sin 17^\circ\ 32'}{2{\cdot}9}$$

$\therefore A = \underline{40^\circ\ 52'}$, since $A$ must be acute.

| *No.* | *Log.* |
|---|---|
| 6·3 | 0·7993 |
| sin 17° 32′ | $\bar{1}{\cdot}4789$ |
| | 0·2782 |
| 2·9 | 0·4624 |
| sin $A$ | $\bar{1}{\cdot}8158$ |

Then $C = 180^\circ - (A + B) = 180^\circ - 58^\circ\ 24' = \underline{121^\circ\ 36'}$.

[The student should note that, having found log sin $A$, which is $\bar{1}{\cdot}8158$, we do not need to find sin $A$; we simply read off the angle $A$ directly from a table of log sines.]

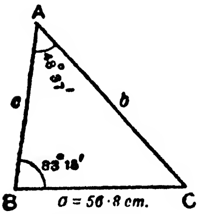



Fig. 270.

## II. Given one side and two angles

The third angle is found from the fact that the sum of the angles is equal to 180°. The two unknown sides are then calculated from the sine rule.

*Example.*—Solve the following triangle:

$a = 56{\cdot}8$ cm., $A = 48^\circ\ 37'$, $B = 83^\circ\ 15'$.

$$C = 180° - (A+B) = 180° - 131° \, 52' = \underline{48° \, 8'}.$$

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\therefore b = \frac{a \sin B}{\sin A}$$

$$= \frac{56 \cdot 8 \sin 83° \, 15'}{\sin 48° \, 37'}$$

$$= \underline{75 \cdot 16 \text{ cm.}}$$

| No. | Log. |
|---|---|
| 56·8 | 1·7543 |
| sin 83° 15′ | $\bar{1}$·9969 |
| | 1·7512 |
| sin 48° 37′ | $\bar{1}$·8752 |
| 75·16 | 1·8760 |

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

$$\therefore c = \frac{a \sin C}{\sin A}$$

$$= \frac{56 \cdot 8 \sin 48° \, 8'}{\sin 48° \, 37'}$$

$$= \underline{56 \cdot 37 \text{ cm.}}$$

| No. | Log. |
|---|---|
| 56·8 | 1·7543 |
| sin 48° 8′ | $\bar{1}$·8720 |
| | 1·6263 |
| sin 48° 37′ | $\bar{1}$·8752 |
| 56·37 | 1·7511 |

**III. Given two sides and the included angle**

The third side is found from the cosine rule, and one of the other angles is then found either from the cosine rule or from the sine rule. Here again, in using the sine rule, in order to avoid ambiguity we find one of the two smaller angles, which we know to be acute. The third angle is the supplement of the sum of the other two.

*Example.*—Solve the triangle for which $b = 119 \cdot 2$ ft., $c = 35 \cdot 6$ ft., $A = 26° \, 16'$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$= 14{,}209 + 1267 - 7610$$

$$= 7866$$

$$a \simeq \underline{88 \cdot 7 \text{ ft.}}$$

| No. | Log. |
|---|---|
| 119·2 | 2·0763 |
| 35·6 | 1·5514 |
| cos 26° 16′ | $\bar{1}$·9526 |
| 3805 | 3·5803 |

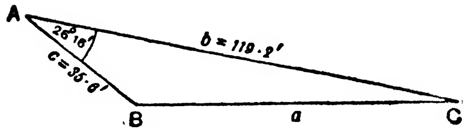



FIG. 271.

The two smaller sides are $c$ and $a$, and hence the two smaller angles are $C$ and $A$. Since $A$ is given we shall calculate $C$.

$$\frac{\sin C}{\sin A} = \frac{c}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$= \frac{35{\cdot}6 \sin 26^\circ\, 16'}{88{\cdot}7}$$

$\therefore$ $\underline{C = 10^\circ\, 14'}$, since $C$ must be acute.

$$\therefore B = 180^\circ - (A + C)$$

$$= 180^\circ - 36^\circ\, 30' = \underline{143^\circ\, 30'}.$$

| *No.* | *Log.* |
|---|---|
| 35·6 | 1·5514 |
| sin 26° 16′ | $\bar{1}$·6460 |
| | 1·1974 |
| 88·7 | 1·9479 |
| sin $C$ | $\bar{1}$·2495 |

## IV. Given two sides and a non-included angle

In this case there may be two possible solutions, i.e. there may be two triangles satisfying the given data. It is usually easy to decide whether there are two solutions or only one by drawing a rough figure. The following examples show how to proceed.

*Example.*—Solve the triangle given that $a = 7{\cdot}2$ in., $b = 11{\cdot}3$ in., $A = 34^\circ$.

If we try to draw the figure we find that there are two possible triangles, viz. $ACB_1$ and $ACB_2$ in Fig. 272.

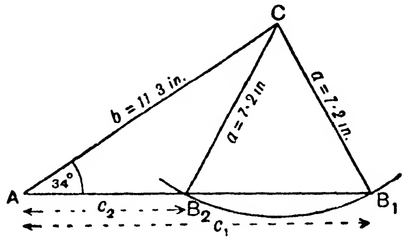



FIG. 272.

$$\frac{\sin B}{\sin A}=\frac{b}{a}$$

$$\therefore\ \sin B=\frac{b\sin A}{a}$$

$$=\frac{11{\cdot}3\sin 34^\circ}{7{\cdot}2}$$

$$\therefore\ B=61^\circ\ 23'$$

| No. | Log. |
|---|---|
| 11·3 | 1·0531 |
| sin 34° | $\bar{1}$·7476 |
| | 0·8007 |
| 7·2 | 0·8573 |
| sin $B$ | $\bar{1}$·9434 |

or $B=180^\circ-61^\circ\ 23'=118^\circ\ 37'$.

In Fig. 272, $\angle CB_1A=61^\circ\ 23'$, $\angle CB_2A=118^\circ\ 37'$.

(i) If $B=61^\circ\ 23'$,

$$C=180^\circ-(A+B)$$
$$=180^\circ-95^\circ\ 23'$$
$$=84^\circ\ 37'.$$

Also $\dfrac{c}{\sin C}=\dfrac{a}{\sin A}$

$$\therefore\ c=\frac{a\sin C}{\sin A}$$

$$=\frac{7{\cdot}2\sin 84^\circ\ 37'}{\sin 34^\circ}$$

$$\simeq 12{\cdot}8$$

| No. | Log. |
|---|---|
| 7·2 | 0·8573 |
| sin 84° 37′ | $\bar{1}$·9981 |
| | 0·8554 |
| sin 34° | $\bar{1}$·7476 |
| 12·81 | 1·1078 |

(ii) If $B = 118° \; 37'$,

$$\begin{aligned} C &= 180° - (A + B) \\ &= 180° - 152° \; 37' \\ &= 27° \; 23'. \end{aligned}$$

Also $\dfrac{c}{\sin C} = \dfrac{a}{\sin A}$

$$\therefore \; c = \frac{a \sin C}{\sin A}$$

$$= \frac{7{\cdot}2 \sin 27° \; 23'}{\sin 34°}$$

$$\simeq 5{\cdot}92$$

| *No.* | *Log.* |
|---|---|
| 7·2 | 0·8573 |
| sin 27° 23′ | $\bar{1}{\cdot}6627$ |
| | 0·5200 |
| sin 34° | $\bar{1}{\cdot}7476$ |
| 5·021 | 0·7724 |

There are therefore two possible solutions, viz. :

$$B = 61° \; 23', \quad C = 84° \; 37', \; c = 12{\cdot}8 \text{ in.},$$

or

$$B = 118° \; 37', \; C = 27° \; 23', \; c = 5{\cdot}92 \text{ in.}$$

*Example.*—Solve the triangle given that $b = 21$ ft., $c = 16$ ft., $B = 50°$.

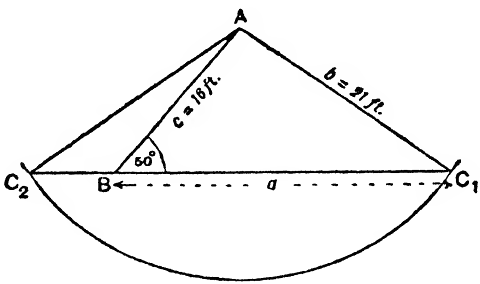



FIG. 273.

Drawing the figure in this case, we find that there is only one possible triangle, viz. the triangle $ABC_1$, in Fig. 273.

$$\frac{\sin C}{\sin B}=\frac{c}{b}$$

$$\therefore \sin C=\frac{c \sin B}{b}$$

$$=\frac{16 \sin 50^\circ}{21}$$

$$\therefore C=35^\circ\ 42'$$

| *No.* | *Log.* |
|---|---|
| 16 | 1·2041 |
| sin 50° | $\bar{1}$·8843 |
| | 1·0884 |
| 21 | 1·3222 |
| sin $C$ | $\bar{1}$·7662 |

or $\qquad C=180^\circ-35^\circ\ 42'=144^\circ\ 18'$

If $C=35^\circ\ 42'$, $A=180^\circ-(B+C)=180^\circ-85^\circ\ 42'=94^\circ\ 18'$.

If $C=144^\circ\ 18'$, $A=180^\circ-(B+C)=180^\circ-194^\circ\ 18'=-14^\circ\ 18'$, which is impossible. Thus the latter value of $C$ is impossible, and there is only one solution.

The fact that there is only one solution will always reveal itself in this way, even if we do not draw the figure.

$$\frac{a}{\sin A}=\frac{b}{\sin B}$$

$$\therefore a=\frac{b \sin A}{\sin B}$$

$$=\frac{21 \sin 94^\circ\ 18'}{\sin 50^\circ}$$

$$=\frac{21 \sin 85^\circ\ 42'}{\sin 50^\circ}$$

$$\simeq 27\cdot 3$$

| *No.* | *Log.* |
|---|---|
| 21 | 1·3222 |
| sin 85° 42′ | $\bar{1}$·9988 |
| | 1·3210 |
| sin 50° | $\bar{1}$·8843 |
| 27·33 | 1·4367 |

There is therefore only one solution, viz.

$$C=35^\circ\ 42',\ A=94^\circ\ 18',\ a=27\cdot 3 \text{ ft.}$$

NOTE.—In both the examples above, the *given* angle is acute. If the *given* angle is *obtuse*, then the angle found from

the sine formula must be acute, since a triangle cannot have two obtuse angles. In that case therefore there is only one solution.

If the *given* angle is *acute*, a comparison of Figs. 272 and 273 shows that there are two solutions or one solution according as the side opposite the given angle is less or greater than the other given side. There is one exception to this rule, viz. when, in Fig. 272, $B_1$ and $B_2$ coincide; in that case there is only one triangle, which is right-angled at $B$.

**Alternative method of solution, by dividing into right-angled triangles**

It might be pointed out that while the methods for solving triangles given above are the most direct, any triangle can be solved without the use of the sine or cosine rule, by dividing it into two right-angled triangles as shown in the examples below.

*Example.*—Solve the triangle for which

$$b = 119{\cdot}2 \text{ ft.}, \; c = 35{\cdot}6 \text{ ft.}, \; A = 26^\circ\ 16'.$$

[Compare p. 323.]

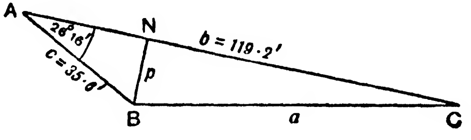



Fig. 274.

Draw $BN$ perpendicular to $AC$.

Then $\triangle$'s $ABN$, $CBN$ are both right-angled triangles.

Let $BN = p$.

$$\frac{p}{35 \cdot 6} = \sin 26° \ 16'$$

$$\therefore p = 35 \cdot 6 \sin 26° \ 16'$$
$$= 15 \cdot 75$$

$$\frac{AN}{p} = \cot 26° \ 16'$$

$$\therefore AN = p \cot 26° \ 16'$$
$$= 15 \cdot 75 \cot 26° \ 16'$$
$$= 31 \cdot 93$$

$$\therefore CN = 119 \cdot 2 - 31 \cdot 93 = 87 \cdot 27$$

Hence

$$\tan C = \frac{p}{CN} = \frac{15 \cdot 75}{87 \cdot 27}$$

$$\therefore \underline{C = 10° \ 14'}.$$

$$\therefore B = 180° - (A + C)$$
$$= 180° - 36° \ 30' = \underline{143° \ 30'}.$$

Also $\frac{a}{p} = \operatorname{cosec} C$

$$\therefore a = p \operatorname{cosec} C = \frac{15 \cdot 75}{\sin 10° \ 14'} = 88 \cdot 67 \simeq \underline{88 \cdot 7 \text{ ft.}}$$

| *No.* | *Log.* |
|---|---|
| 35·6 | 1·5514 |
| sin 26° 16′ | $\bar{1}$·6460 |
| $p$ = 15·75 | 1·1974 |
| 15·75 | 1·1974 |
| cot 26° 16′ | 0·3067 |
| $AN$ = 31·93 | 1·5041 |
| 15·75 | 1·1974 |
| 87·27 | 1·9408 |
| tan $C$ | $\bar{1}$·2566 |
| 15·75 | 1·1974 |
| sin 10° 14′ | $\bar{1}$·2496 |
| $a$ = 88·67 | 1·9478 |

*Example.*—Solve the following triangle: $a = 56 \cdot 8$ cm., $A = 48° \ 37'$, $B = 83° \ 15'$.

[Compare p. 322.]

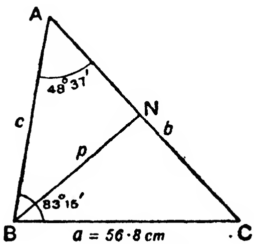



FIG. 275.

Draw $BN$ perpendicular to $AC$; let $BN = p$.

$$C = 180° - (A + B)$$
$$= 180° - 131° \, 52' = \underline{48° \, 8'}$$

$$\frac{p}{56{\cdot}8} = \sin C$$

$$\therefore p = 56{\cdot}8 \sin C = 56{\cdot}8 \sin 48° \, 8'$$
$$= 42{\cdot}30$$

$$\frac{CN}{a} = \cos C$$

$$\therefore CN = a \cos C = 56{\cdot}8 \cos 48° \, 8'$$
$$= 37{\cdot}90$$

Also $\frac{AN}{p} = \cot 48° \, 37'$

$$\therefore AN = 42{\cdot}30 \cot 48° \, 37'$$
$$= 42 \, 30 \tan 41° \, 23'$$
$$= 37{\cdot}27$$

$$\therefore b = AC = AN + CN$$
$$= 37{\cdot}27 + 37{\cdot}90$$
$$= \underline{75{\cdot}17 \text{ cm.}}$$

Finally $\frac{c}{p} = \operatorname{cosec} 48° \, 37'$

$$\therefore c = 42{\cdot}30 \operatorname{cosec} 48° \, 37'$$
$$= \frac{42{\cdot}30}{\sin 48° \, 37'} = \underline{56{\cdot}37 \text{ cm.}}$$

| *No.* | *Log.* |
|---|---|
| 56·8 | 1·7543 |
| sin 48° 8′ | $\bar{1}$·8720 |
| $p$ = 42·30 | 1·6263 |
| 56·8 | 1·7543 |
| cos 48° 8′ | $\bar{1}$·8244 |
| 37·90 | 1·5787 |
| 42·30 | 1·6263 |
| tan 41° 23′ | $\bar{1}$·9450 |
| 37·27 | 1·5713 |
| 42·30 | 1·6263 |
| sin 48° 37′ | $\bar{1}$·8752 |
| 56·37 | 1·7511 |

## Exercise XLI

**Solve the following triangles by calculation. Also draw the triangles accurately to scale and measure the remaining sides and angles, and compare the results with those obtained by calculation.**

**1.** $a = 6{\cdot}7$ cm., $b = 6$ cm., $c = 4$ cm.

**2.** $a = 19$ yd., $b = 48$ yd., $c = 36$ yd.

3. $b=3$ in., $A=32°$, $C=74°$.

4. $c=28$ ft., $B=45°$, $C=106°$.

5. $a=2{\cdot}3$ cm., $b=3{\cdot}6$ cm., $C=60°$.

6. $a=101$ ml., $c=78$ ml., $B=11°\ 25'$.

7. $b=3{\cdot}5$ in., $c=4$ in., $B=52°$.

8. $a=6$ cm., $b=5$ cm., $A=77°\ 42'$.

9. $a=1$ ft., $c=2$ ft., $C=135°$.

10. Find, by calculation, the remaining sides and angles in the triangles in Fig. 276.

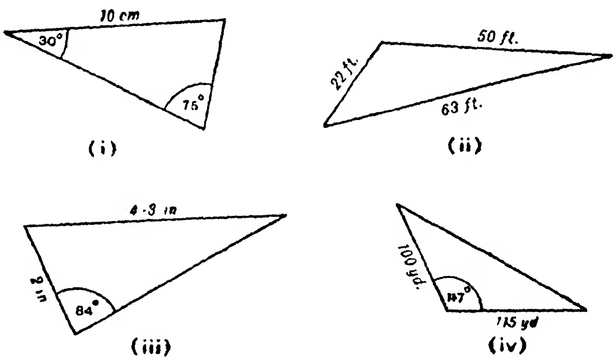



Fig. 276.

11. If $A=62°$, $B=37°\ 30'$ and $b=40$ ft., find $c$.

12. Find the area of the triangle $DEF$ in which $DE=1{\cdot}6$ in., $E\widehat{D}F=35°$ and $D\widehat{F}E=124°$.

13. In a triangle $PQR$, $PQ=15$ cm., $PR=17$ cm. and $\angle P=43°\ 30'$; find the angle $Q$.

14. $ABC$ is a triangle of area 18 sq. in., and $AB=5$ in., $BC=9$ in. Find the angle $ACB$.

15. $DEF$ is a triangle of area 41 sq. yd., and $\angle D=17°$, $DE=15$ yd. Find the lengths of the other sides.

16. A weight is suspended from a ceiling by two chains, of lengths 2 ft. and 4 ft., attached to points in the ceiling 5 ft. apart. Find the distance of the weight below the ceiling.

**17.** A mechanism, shown in Fig. 277, consists of two pistons which slide in cylinders whose axes are inclined to each other at 120°, and a connecting-rod 3 ft. long. Find $OB$ when $OA = 2$ft.

If the least and greatest distances of $A$ from $O$ are 2 ft. and 2 ft. 6 in. find the travel of the piston $B$.

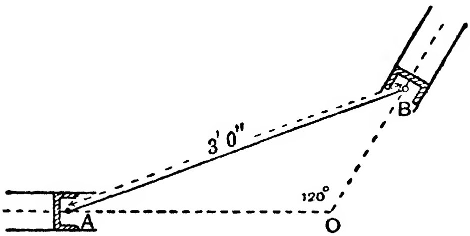



FIG. 277.

**18.** The mid-points of the legs of a step-ladder are connected by a rope 3 ft. 3 in. long. When the ladder is fully opened out the legs make angles 54° and 73° with the horizontal. Find the length of each leg of the ladder.

## Harder problems

*Example.*—A surveyor, using a theodolite mounted at the top of a stake $AC$ which is 4 ft. 9 in. high, observes the angles of elevation of two marks $D$ and $E$ on a vertical pole at $B$ to be 12° 40′ and 9° 15′ (Fig. 278). The marks are 2 ft. 6 in. apart, and the lower one is 3 ft. 0 in. above the ground. Find the height of $B$ above $A$.

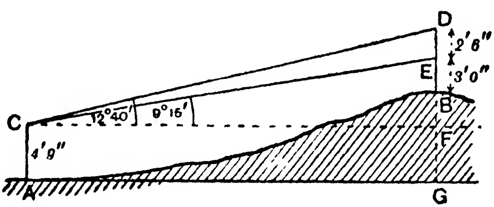



FIG. 278.

Draw $CF$ and $AG$ horizontal, cutting the vertical through $B$ in $F$ and $G$. Then $FG = CA = 4$ ft. 9 in.

Required height

$= BG = BF + 4$ ft. 9 in. $= (EF - 3$ ft. 0 in.$) + 4$ ft. 9 in.

$= EF + 1$ ft. 9 in.

Now $EF = EC \sin 9° 15'$, and we can find $EC$ by applying the sine rule to $\triangle CDE$.

For $\quad C\widehat{D}E = 90° - 12° 40' = 77° 20'$,

and $\quad D\widehat{C}E = 12° 40' - 9° 15' = 3° 25'$.

$$\therefore \frac{EC}{\sin 77° 20'} = \frac{DE}{\sin 3° 25'} = \frac{2{\cdot}5}{\sin 3° 25'}$$

$$\therefore EC = \frac{2{\cdot}5 \sin 77° 20'}{\sin 3° 25'}$$

$$\therefore EF = EC \sin 9° 15'$$

$$= \frac{2{\cdot}5 \sin 77° 20' \,.\, \sin 9° 15'}{\sin 3° 25'}$$

$$\simeq 6{\cdot}58 \text{ ft.}$$

$$= 6 \text{ ft. } 7 \text{ in.}$$

$\therefore$ Height of $B$ above $A$

$= 6$ ft. 7 in. $+ 1$ ft. 9 in.

$= 8$ ft. 4 in.

| *No.* | *Log.* |
|---|---|
| 2·5 | 0·3979 |
| sin 77° 20′ | $\bar{1}{\cdot}9893$ |
| sin 9° 15′ | $\bar{1}{\cdot}2061$ |
| | $\bar{1}{\cdot}5933$ |
| sin 3° 25′ | $\bar{2}{\cdot}7752$ |
| 6·579 | 0·8181 |

*Example.*—The roof truss represented in Fig. 279 has the dimensions shown. Find the length of $AE$.

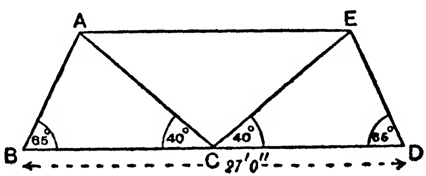



Fig. 279.

From symmetry $C$ is the mid-point of $BD$.

$$\therefore BC = 13 \text{ ft. } 6 \text{ in.}$$

$$B\widehat{A}C = 180^\circ - (65^\circ + 40^\circ)$$

$$= 75^\circ$$

$$\therefore \text{In } \triangle ABC, \quad \frac{AC}{\sin 65^\circ} = \frac{BC}{\sin 75^\circ} = \frac{13{\cdot}5}{\sin 75^\circ}$$

$$\therefore AC = \frac{13{\cdot}5 \sin 65^\circ}{\sin 75^\circ}$$

Since $AE$ is parallel to $BC$, $E\widehat{A}C = A\widehat{C}B = 40^\circ$.

$\therefore$ In the isosceles $\triangle ACE$,

$$AE = 2AC \cos E\widehat{A}C$$

$$= 2 \times \frac{13{\cdot}5 \sin 65^\circ}{\sin 75^\circ} \times \cos 40^\circ$$

$$= \frac{27 \sin 65^\circ \cos 40^\circ}{\sin 75^\circ}$$

$$= 19{\cdot}41 \text{ ft.}$$

| *No.* | *Log.* |
|---|---|
| 27 | 1·4314 |
| sin 65° | $\bar{1}$·9573 |
| cos 40° | $\bar{1}$·8843 |
| | 1·2730 |
| sin 75° | $\bar{1}$·9849 |
| 19·41 | 1·2881 |

NOTE.—In this example the known length, $BC$, and the required length, $AE$, are sides of different triangles, so that we cannot apply the sine rule directly. We therefore use the common side of those two triangles, viz. $AC$, as a " link " to pass from one triangle to the other.

*Example.*—An aeroplane is sighted simultaneously from two stations $A$ and $B$, $B$ being one mile north-east of $A$. To the observer at $A$ the aeroplane appears due north at an elevation of 34° ; to the observer at $B$ it appears in a direction N. 72° W. Find the height of the aeroplane.

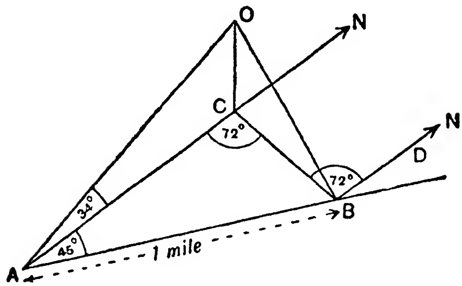



FIG. 280.

In Fig. 280 $O$ represents the aeroplane and $C$ is the point on the ground vertically below $O$.

$$\widehat{ACB} = \text{alternate angle } \widehat{CBD} = 72°.$$

$$\therefore \widehat{ABC} = 180° - (45° + 72°) = 63°.$$

$$\therefore \text{ In } \triangle ABC, \frac{AC}{\sin 63°} = \frac{AB}{\sin 72°} = \frac{1}{\sin 72°}$$

$$\therefore AC = \frac{\sin 63°}{\sin 72°}.$$

In $\triangle ACO$, the angle at $C$ is a right angle,

$$\therefore \frac{OC}{AC} = \tan 34°$$

$$\therefore OC = AC \tan 34°$$

$$= \frac{\sin 63° \tan 34°}{\sin 72°}$$

$$= 0{\cdot}632 \text{ mile}$$

$$= 0{\cdot}632 \times 5280 \text{ ft.}$$

$$= 3337 \text{ ft.}$$

| No. | Log. |
|---|---|
| sin 63° | $\bar{1}{\cdot}9499$ |
| tan 34° | $\bar{1}{\cdot}8290$ |
| | $\bar{1}{\cdot}7789$ |
| sin 72° | $\bar{1}{\cdot}9782$ |
| 0·6320 | $\bar{1}{\cdot}8007$ |

## Exercise XLII

**1.** A ship sights two lighthouses both due North. After steaming 5 miles due East their bearings are N. 50° W. and N. 35° W. Find the distance between the lighthouses.

**2.** A rod 6 in. long rests at an inclination of 18° to the horizontal across a groove formed by two planes each inclined at 54° to the horizontal (Fig. 281). Find the height of the lower end of the rod above the bottom of the groove.

**3.** A tower stands on a hillside which may be regarded as a plane of inclination 15° to the horizontal. The elevation of the top of the tower from one point on the hillside is 20°, and from a point 100 yd. farther up the hill towards the tower the elevation is 28°. Find the height of the tower.

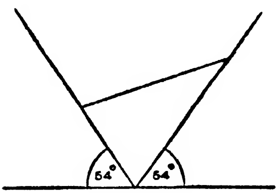



Fig. 281.

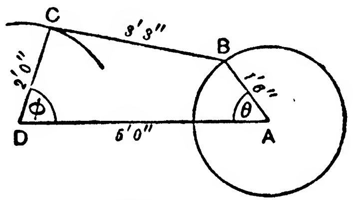



Fig. 282.

**4.** The dimensions of a four-bar linkage are shown in Fig. 282. Find the angle $\phi$ when $\theta = 45°$.

[Hint.—Join *BD* and find the angles *BDA*, *BDC* separately.]

**5.** The elevation of the top of a wireless mast from one window of a house is $\alpha$. From a window a distance $h$ higher up, in the same vertical line, the elevation is $\beta$. Show that the height of the top of the mast above the level of the first window is $h\dfrac{\sin\alpha\cos\beta}{\sin(\alpha-\beta)}$.

Calculate this height when $h = 20$ ft., $\alpha = 30°$, $\beta = 22°$.

**6.** Find the lengths of the three unmarked members of the framework in Fig. 283.

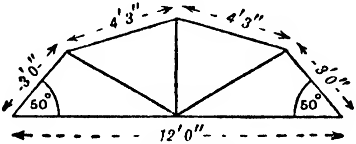



Fig. 283.

**7.** To find the distance of a landmark $O$ from a point $A$ from which it is invisible, a surveyor takes readings from two points $B$ and $C$ from which it is visible. His measurements are recorded in Fig. 284. Find the distance of $O$ from $A$.

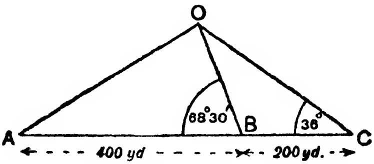



FIG. 284

**8.** A man at $A$ wishes to find the distance between two points $P$ and $Q$ (Fig. 285) on the opposite side of a river which he cannot cross. He walks a measured distance to a point $B$ and observes the angles $BAP$, $BAQ$, $ABP$, $ABQ$. Show how he can calculate the distance $PQ$ from his measurements.

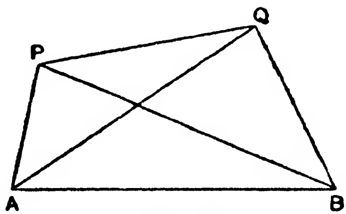



FIG. 285.

**9.** A steamer is observed from the top of a cliff 530 ft. high to bear S. 23° W. at an angle of depression 4° 30′. Four minutes later it is observed in a direction S. 41° E. and the angle of depression is then 6° 12′. Find the course and speed of the ship (assumed constant).

**10.** A man making a geographical survey wishes to find the height of a point $B$ above his station at $A$, but the ground between $A$ and $B$ is impassable. He therefore chooses a second station $C$ and makes the following measurements: elevations of $B$ and $C$ from $A$, distance $AC$, bearings of $B$ from $A$ and $C$. Show how he can calculate the required height from these data.

**11.** Use the sine rule to prove the theorem on p. 128, viz. that the bisectors of an angle of a triangle divide the opposite side in

the ratio of the sides containing the angle. [Hint.—Use $AX$, or $AY$ (Fig. 52), as a link between the triangles containing the required sides.]

## Miscellaneous Exercise XLIII

**1.** A vertical mast 50 ft. high stands on a hillside, of inclination 12°, facing North. Find, to the nearest inch, the length of its shadow when the Sun is due South at an elevation 81°.

**2.** A ship steaming due East is seen 6 miles away in the direction N. 50° W. Ten minutes later it is observed in the direction N. 30° W. Find its speed.

**3.** The floor of a hall is octagonal in shape and its area is 3000 sq. ft. Find the length of each side.

**4.** A man wishing to go from $A$ to $B$ can either cycle along a straight path which runs direct from $A$ to $B$ or go by car along a main road which runs in two straight parts $AC$, $CB$ (Fig. 286). If he can cycle at 8 m.p.h. and his average speed by car is 25 m.p.h., which is the quicker way?

**5.** $P$, $Q$ are two landmarks on a straight road, $P$ being $\frac{1}{4}$ mile due West of $Q$. A house bears S. 67° E. from $P$, and S. 42° W. from $Q$; find, to the nearest yard, its distance from the road.

**6.** Find, to the nearest inch, the lengths of the members $AD$ and $BD$ in the framed structure shown in Fig. 287.

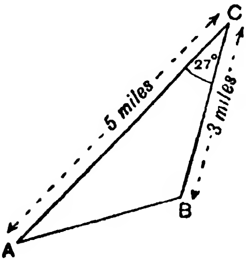



Fig. 286.

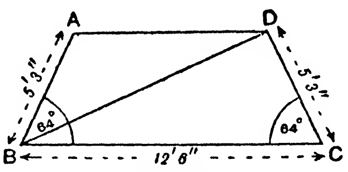



Fig. 287.

**7.** A cylindrical boiler, 12 ft. 0 in. long and 5 ft. 0 in. in diameter, has two furnace tubes, each 1 ft. 3 in. in diameter, running through it (Fig. 288). Find the volume of water (in gallons) required to fill the boiler to a depth of 4 ft. 0 in. [1 cu. ft. ≏ 6·23 gallons.]

8. Two ships leave the same port at the same time. One steams at 12 knots in the direction N. 60° E., the other at 15 knots in the direction S. 75° E. How far apart are they 3 hours later, and what is then the bearing of the second ship from the first ? [1 knot = a speed of 1 nautical mile per hour ; 1 nautical mile $\doteqdot$ 1·15 statute miles.]

9. A circular railway track is to be constructed so as to pass through three points $A$, $B$, $C$. If $BC = 64$ yd., $CA = 35$ yd., $AB = 82$ yd., find the radius of the circle.

[Hint.—If $O$ is the centre of the circle circumscribing a triangle $ABC$, and $ON$ is perpendicular to $BC$ (Fig. 289),

$$B\widehat{O}N = \tfrac{1}{2}\, B\widehat{O}C = \tfrac{1}{2}(2B\widehat{A}C) = B\widehat{A}C.$$

$$\therefore\ \frac{BN}{BO} = \sin B\widehat{O}N = \sin A.$$

$$\therefore\ \text{Radius} = BO = \frac{BN}{\sin A} = \frac{a}{2\sin A}.\Big]$$

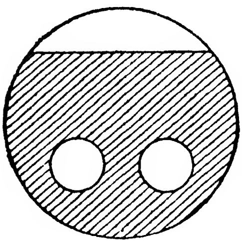


FIG. 288.

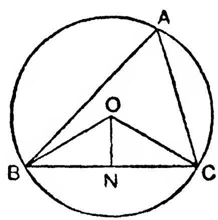



FIG. 289.

10. Two anti-aircraft batteries, $P$ and $Q$, are six miles apart, $Q$ being due North of $P$. An aeroplane, flying horizonally, is observed from $P$ at an elevation of 40° due North, and at the same instant from $Q$ at an elevation of 27° South. One minute later it is observed from $Q$ at an elevation of 70° due South. Find the speed of the aeroplane.

11. Solve the triangle $ABC$ given that $B = 30°$, $a = 4{\cdot}4$ in. $b + c = 6{\cdot}5$ in.

[Hint.—If $BA$ is produced to $D$ so that $AD = AC$ (Fig. 290), then $BD = b + c$. The angle $D$ can now be calculated, and $\widehat{A} = 2\widehat{D}$.]

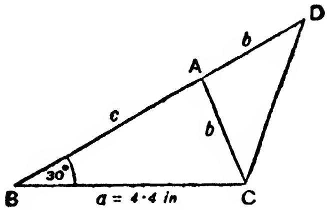



Fig. 290.

**12.** In the "quick-return motion" shown in Fig. 291, $PQ$ is a rod 1 ft. 6 in. long whose end $P$ describes a circle of radius 3 in., while its end $Q$ moves along a straight guide distant 10 in. from the centre of the circle. Find $x$ when $\theta = 130°$.

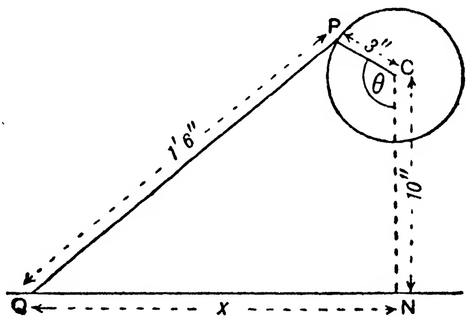



Fig. 291.

# CALCULUS

## CHAPTER XV

### DIFFERENTIATION

One of the most powerful methods in modern mathematics is that of the calculus, the ideas of which were conceived by Archimedes in the third century B.C. By dividing a segment of a parabola into thin strips and adding together their areas, he found an approximation to the area of the segment. He then obtained closer and closer approximations by taking more and thinner strips. By this method of exhaustion he found the area of the segment exactly.

About 1586 Stevinus of Bruges used a method similar to that of Archimedes to find the thrust of a liquid on a surface, and a little later a Jesuit priest, Cavalieri, extended the method to find the volumes of solids. The methods used by these mathematicians to find the whole area, or volume, by dividing it up into small parts is now called "integration," i.e. finding the whole.

In the beginning of the seventeenth century the French mathematician Fermat considered the ratio of infinitesimally small increments and so laid the foundation on which Newton (1642–1727) and Leibniz (1646–1716) later built the theory of "differentiation" or finding rates of change from the ratios of small differences. It was due to the genius of Newton and Leibniz that a great advance was made. Newton conceived the idea of continuous change, and rate of change at an instant or flux, and he described his new subject as "fluxions."

He found that his knowledge of rates of change could be applied to calculate areas and volumes, that is to perform integrations, much more easily than by the method of exhaustion described above. Leibniz discovered the method of differentiation about the same time and we are specially indebted to him for his notation, which is essentially that now in general use.

In the last two centuries calculus has been developed to such an extent that it is now used to deal with problems in every branch of technical science.

## Tank filling at a constant rate

Suppose water starts to flow into an empty tank at a constant rate of 2 cu. ft. per min.; then after $t$ min. the volume of water in the tank is $2t$ cu. ft. If we call this volume V cu. ft., $V=2t$. The graph of this equation is a straight line through the origin of gradient 2. Thus, the rate at which the water is flowing in, or in other words, the rate of increase of V, is equal to the gradient of the graph of V against $t$. In the same way, if the tank had 3 cu. ft. in it when the water started to

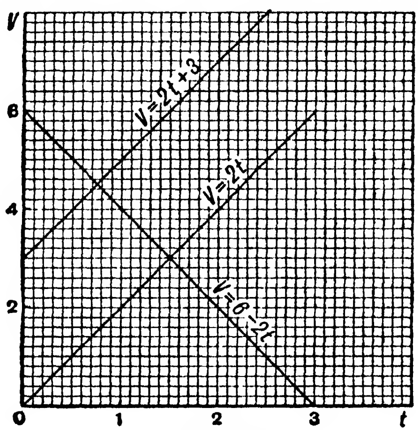



FIG. 292.

flow in, the volume after $t$ min. would be given by $V=2t+3$. This equation also has a straight line graph of gradient 2. Thus, we see that, so long as the water flows in at a constant rate, the graph of V against $t$ is a straight line, and the rate of flow, or the rate of increase of V is the gradient of the line. The graphs of $V=2t$ and $V=2t+3$ are shown in Fig. 292.

## Tank emptying at a constant rate

If a tank has 6 cu. ft. of water in it and water flows out at 2 cu. ft. per min., the volume V cu. ft. after $t$ sec. is given by $V=6-2t$. Thus the graph of V is a straight line, but one with a negative gradient $-2$. Hence, if we regard a rate of decrease of V of 2 cu. ft. per min. as the same as a rate of increase of V of $-2$ cu. ft. per min., it is still true that the rate of increase of V is the gradient of the graph of V against $t$. The graph of $V=6-2t$ is shown in Fig. 292.

We conclude that, if the relationship between V and $t$ is $V=at+b$, V is increasing at a constant rate $a$.

## Tank filling and emptying at a variable rate

The following table gives the volume of water, V cu. ft., in a tank, which was filled at a variable rate, $t$ min. after the water began to flow in. After a time a plug was pulled out and the tank began to empty.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ min. .. | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{2}$ | 4 |
| V cu. ft. | 2 | 5·50 | 8·90 | 11·15 | 12·05 | 11·00 | 8·60 | 5·45 | 1·20 |

Fig. 293 shows the graph of V against $t$. From $t=1$ to $t=1\frac{1}{2}$, V increased by $(11{\cdot}15-8{\cdot}90)=2{\cdot}25$, so that, if the water had flowed in at a constant rate during this time, it would have flowed in at $\dfrac{2{\cdot}25}{\frac{1}{2}}=4{\cdot}5$ cu. ft. per min. If the water had flowed in at a constant rate the graph of V would have been the chord AB instead of the curve from A to B, and the constant rate of flow, 4·5 cu. ft. per min., is the gradient of the chord AB. *This constant rate of flow is called the average*

*(or mean) rate of increase of V in the time from* t = 1 *to* t = $1\frac{1}{2}$. Similarly, we find that, if we take an interval of time from $t=1$ to $t=1\frac{1}{8}$, V increases by $(9{\cdot}60-8{\cdot}90)=0{\cdot}70$, and for this increase the water would have had to flow at a constant rate of $\dfrac{0{\cdot}70}{\frac{1}{8}}=5{\cdot}60$ cu. ft. per min., or the gradient of the chord AC. Again, for a still smaller interval from $t=1$ to $t=1{\cdot}05$, the

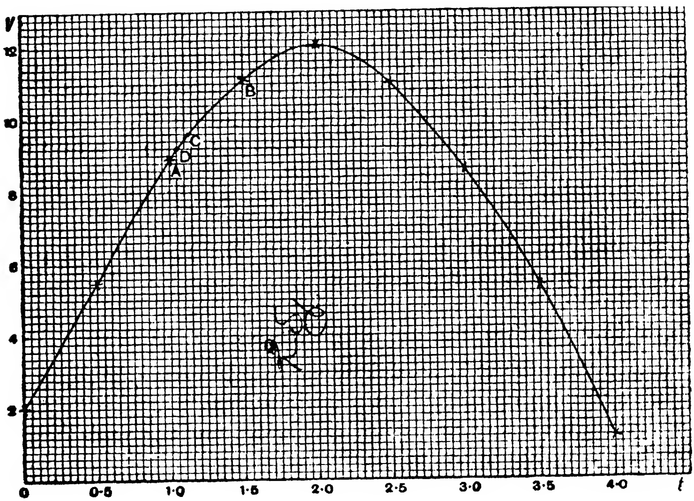



FIG. 293.

water would have had to flow at a constant rate of $\dfrac{9{\cdot}20-8{\cdot}90}{0{\cdot}05}$ $=6{\cdot}00$ cu. ft. per min., or the gradient of the chord AD. Now as we take shorter and shorter intervals of time we get chords which are closer and closer to the tangent at A and the corresponding constant rates of increase of V approach the gradient of the tangent at A. For this reason we call the gradient of the tangent at A *the rate of increase of V at the instant* t = 1. Notice that it is a rate which can never

be measured exactly by using a stop-watch; it can only be approximated to by taking a very short interval of time, such as $\frac{1}{20}$th sec.

Therefore *the rate of increase of V at any value of* t

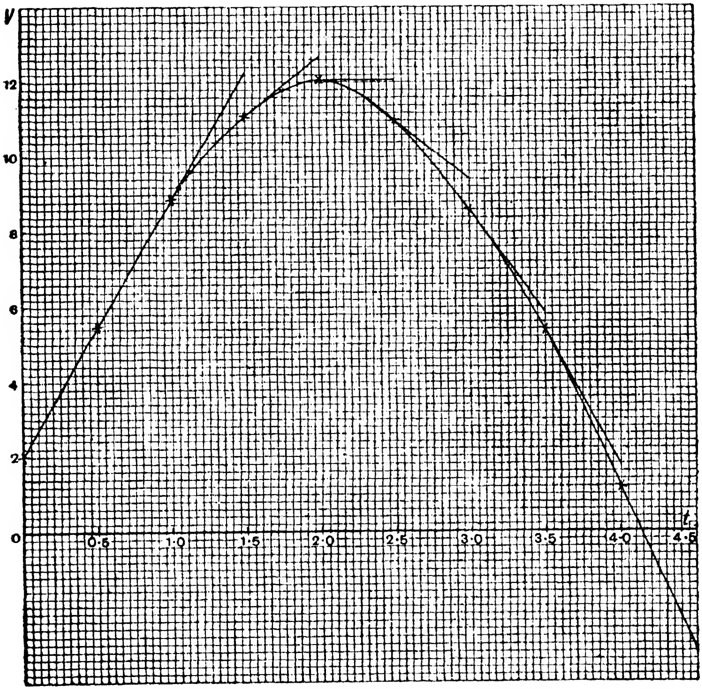



FIG. 294.

*is the gradient of the tangent to the graph of V against* t *at the point given by that value of* t. The rate of increase of V at the instant when the time is $t$ may be denoted by $\dot{V}$; this notation was introduced by Newton.

In Fig. 294 the graph of Fig. 293 is repeated and the tangents at $t=0$, $\frac{1}{2}$, 1, etc., are drawn. The increase of V along each

tangent during $\frac{1}{2}$ min. is read from the graph and entered in the second line of the table below. The third line of the table shows the values of $\dot{V}$, which are the rates of increase of V, and are obtained by dividing the increments of V in the second line by $\frac{1}{2}$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ min. .. | 0 | 0·5 | 1·0 | 1·5 | 2·0 | 2·5 | 3·0 | 3·5 | 4·0 |
| Increase of V along tangent in $\frac{1}{2}$ min. .. | 3·5 | 3·5 | 3·5 | 1·55 | 0 | −1·6 | −2·7 | −3·6 | −4·8 |
| $\dot{V}\left(\frac{\text{cu. ft.}}{\text{min.}}\right)$ .. | 7·0 | 7·0 | 7·0 | 3·1 | 0 | −3·2 | −5·4 | −7·2 | −9·6 |

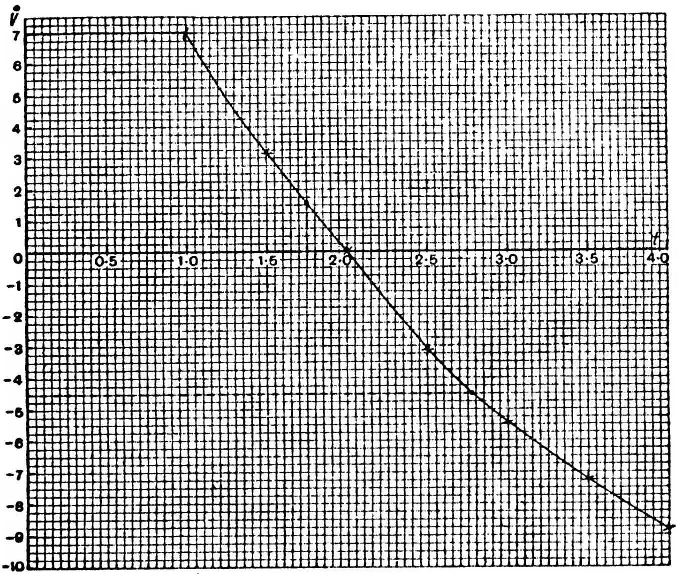



Fig. 295.

After $t=2$, V decreases and so its rate of increase and also the gradient of the tangent are negative.

Fig. 295 shows the graph of $\dot{V}$ plotted from this table. From this graph we can read the rate of increase of V at any

instant, and find at what instant it is increasing or decreasing at a given rate. For example, at $t=1\frac{3}{4}$, $\dot{V}=1{\cdot}5$, so the water is flowing in at 1·5 cu. ft. per min. Again, the water is flowing out at 4·5 cu. ft. per min. when $\dot{V}=-4{\cdot}5$, which is at $t=2{\cdot}8$.

The graph of $\dot{V}$ is sometimes called the *derived curve* of the graph of V.

### Difference ratio

Let the volume in the tank be V cu. ft. after $t$ sec., and V′ cu. ft. after $t'$ sec. In Fig. 296 P and Q are the points on the graph given by $t$ and $t'$, and hence $PK=MN=t'-t$

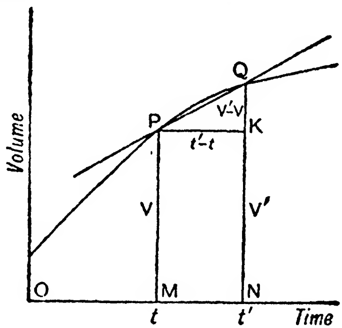



FIG. 296.

and $KQ=NQ-MP=V'-V$. As the time increases by $t'-t$, or PK, the volume increases by $V'-V$, or KQ.

∴ Average rate of increase of the volume in the time $t'-t$

$$=\frac{V'-V}{t'-t}=\frac{KQ}{PK}=\text{gradient of the chord PQ.}$$

$\dfrac{V'-V}{t'-t}$ may be called a difference ratio because it is the ratio of the differences of the volumes to the difference of the times.

The increase in the time from $t$ to $t'$ is written $\delta t$, where

the Greek letter $\delta$ is used as shorthand for the words "the increase of." Thus,

$$t' - t = \delta t = \text{the increase of } t.$$

Notice that $\delta t$ does not mean $\delta \times t$; the $\delta$ cannot be separated from the letter that follows it, just as in a trigonometrical ratio like sin 20° the word *sin* cannot be separated from the 20°, being meaningless by itself.

In just the same way

$$V' - V = \delta V = \text{the increase of V.}$$

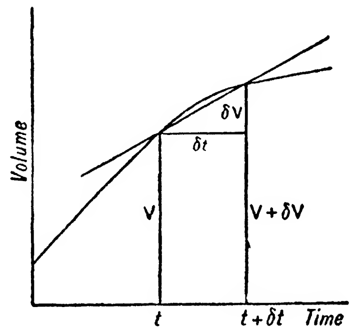



FIG. 297.

Fig. 297 shows the same curve as Fig. 296, but $t'$ and $V'$ are replaced by $t + \delta t$ and $V + \delta V$ respectively.

$$\therefore \text{ the difference ratio } \frac{V' - V}{t' - t} = \frac{\delta V}{\delta t}.$$

Now if we take the time $t'$ closer and closer to the time $t$, that is take $\delta t$ smaller and smaller, the difference ratio (as we have already seen on p. 344) approaches closer and closer to $\dot{V}$, the rate of increase of V at the instant when the time is $t$, or the gradient of the tangent at P.

Leibniz suggested that, because the rate of increase of V at the instant $t$ is obtained from the difference ratio $\frac{\delta V}{\delta t}$ by taking

$\delta t$ smaller and smaller, it should be denoted by a similar notation, and so he wrote $\frac{dV}{dt}$ for $\dot{V}$. Hence,

$$\frac{dV}{dt} = \dot{V} = \text{the rate of increase of V at the instant } t.$$

The notation $\frac{dV}{dt}$ is used more frequently than the dot notation of Newton.

Because $\frac{dV}{dt}$ is obtained from a difference ratio, when we find $\frac{dV}{dt}$ from V we are said to "differentiate" V, and, for reasons which we shall see in the next volume, $\frac{dV}{dt}$ is called "the differential coefficient of V with respect to $t$" or "the derivative of V with respect to $t$."

### Velocity and acceleration

If a body moves a distance $s$ in a straight line in a time $t$, its velocity at the instant of time given by $t$ is the rate of increase of $s$, or in symbols, $v = \dot{s}$ or $\frac{ds}{dt}$.

The graph obtained by plotting $s$ against $t$ is called the space-time or distance-time graph, and since $v = \frac{ds}{dt}$ the velocity at time $t$ is the gradient of the tangent of the space-time graph at the time $t$.

The acceleration of the body is the rate of increase of its velocity, namely $\dot{v}$ or $\frac{dv}{dt}$. This is the gradient of the tangent to the graph of $v$ against $t$, which is called the velocity-time graph.

*Example.*—If a body starts at a velocity of $u$ ft./sec. and its velocity increases at a uniform acceleration of $f$ ft./sec.$^2$, find the velocity after $t$ sec. What type of curve is the velocity-time graph ?

Since the velocity increases $f$ ft./sec. every second, it increases $ft$. ft./sec. in $t$ sec.

$$\therefore\ v = u + ft.$$

The velocity-time graph is the graph of this equation, which is a straight line of gradient $f$.

*Example.*—The lift $s$ in. of a cam follower after $t$ sec. during one revolution of the cam is given by the following table.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ .. | 0 | 0·04 | 0·08 | 0·12 | 0·16 | 0·20 | 0·24 | 0·28 | 0·32 |
| $s$ .. | 0 | 0·072 | 0·372 | 0·912 | 1·488 | 0·912 | 0·372 | 0·072 | 0 |

Draw a space time graph for the motion of the follower, and from it draw graphs showing the velocity in ft./sec. and the acceleration in ft./sec.$^2$.

Fig. 298 ($a$) shows the space-time graph. From this graph by measuring gradients the following table is constructed.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t$ .. .. | 0 | 0·04 | 0·08 | 0·12 | 0·16 | 0·20 | 0·24 | 0·28 | 0·32 |
| $v$ (in./sec.) | 0 | 3·9 | 11·1 | 21·0 | 0 | −21·0 | −11·1 | −3·9 | 0 |
| $v$ (ft./sec.) | 0 | 0·325 | 0·925 | 1·75 | 0 | −1·75 | −0·925 | −0·325 | 0 |

From this table the velocity time graph in Fig. 298 ($b$) is drawn. The acceleration time graph in Fig. 298 ($c$) is constructed in a similar way. Fig. 298 ($b$) and ($c$) show that the follower starts with an acceleration of 7·5 ft./sec.$^2$, moves with increasing acceleration until $t = 0·1$ ; its acceleration then diminishes but its velocity still increases reaching its greatest value 1·75 ft./sec. at $t = 0·12$. The curve in Fig. 298 ($b$) is the derived curve of the curve in Fig. 298 ($a$), and the curve in Fig. 298 ($c$) the derived curve of the curve in Fig. 298 ($b$).

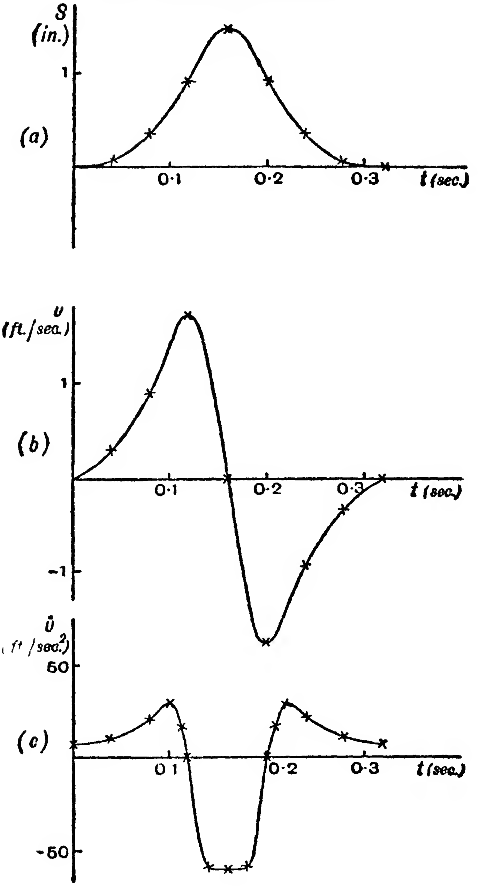



FIG. 298.

## Exercise XLIV

1. A racing motor car attains its full speed from rest in 48 sec. and then covers a measured mile in 12 sec. Find in miles per hour (*a*) its speed over the measured mile, (*b*) its average speed in the first 48 sec., (*c*) its average speed in the first 60 sec.

2. When some water is being boiled its temperature rises from 15° C. to 100° C. in 10 min.; find the average rate of increase of temperature in this time.

3. If a tank contains 40 gallons of water and the water runs out at 4 gallons per minute, find the number of gallons ($n$) left in the tank after $t$ sec. Draw the graph of $n$ against $t$. Express the rate of decrease of $n$ as a negative rate of increase.

4. The following table gives the space $s$ ft. passed over by a projectile in the bore of a gun in $t$ sec.

| | | | | | | |
|---|---|---|---|---|---|---|
| $s$ .. | 0 | 1·2 | 3·6 | 7·0 | 11·3 | 25 |
| $t$ .. | 0 | 0·002 | 0·004 | 0·006 | 0·008 | 0·01 |

Draw a graph of $s$ against $t$, and from it deduce the graph of $\dot{s}$ against $t$. State the muzzle velocity of the gun.

5. The following table gives the height in inches of two boys of the same age at various ages.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Age .. .. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Height of A .. | 31·1 | 34·8 | 37·7 | 40·0 | 42·3 | 44·9 | 47·0 | 49·5 | 51·8 | 54·2 | 56·5 | 58·2 |
| Height of B .. | 30·9 | 33·0 | 35·0 | 37·5 | 40·0 | 44·0 | 49·0 | 52·7 | 55·3 | 56·7 | 58 | 59 |

Using the same axes draw graphs to show the heights of both boys. Show that the height of one boy increases very nearly at a uniform rate and find this rate. Estimate approximately ($a$) during what time the other boy is growing faster than this boy, ($b$) when he is growing fastest, ($c$) his fastest rate of growth.

6. The relation between the distance and time for an electric tramcar starting from rest is given by:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Time in sec. .. | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 |
| Distance in ft. | 0 | 41 | 170 | 410 | 680 | 905 | 1070 | 1195 |

Draw a distance-time graph, and from it deduce the velocity-time and acceleration-time graphs (first and second derived curves). Estimate the velocity and acceleration after 45 sec.

7. If a man's height is $x$ in. after $t$ years and increases $\delta x$ in. in a further time $\delta t$ year, at what average rate has his height increased in the time $\delta t$? How would you express the rate of increase of his height when he is exactly $t$ years old?

8. The velocity of a train in the first 90 sec. of its motion is given by:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ ft./sec. .. | 0 | 6 | 12·0 | 27·7 | 40·5 | 47·2 | 49·5 | 44·2 | 35·2 | 30 | 27·7 |
| $t$ sec. .. | 0 | 5 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 |

Plot a graph of $v$ against $t$, and deduce from it the graph of $\frac{dv}{dt}$, which is the acceleration-time graph. At what time is the acceleration zero, and at what time has the train (*a*) an acceleration 1·1 ft./sec.$^2$, (*b*) a retardation 0·4 ft./sec.$^2$ ?

**9.** When a condenser is being charged the charge on a plate, Q coulombs, increases with the time $t$ sec. according to the following table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ .. | 0 | 0·01 | 0·02 | 0·03 | 0·04 | 0·05 | 0·06 |
| Q .. | 0 | 0·04 | 0·0625 | 0·0775 | 0·0865 | 0·0920 | 0·0962 |

Draw a graph of Q against $t$, and assuming that the charging current $i$ ampères is given by $i=\frac{dQ}{dt}$, draw a graph of $i$ against $t$. What is the greatest value of $i$ ?

**10.** T° C. is the temperature of a body which has been cooling for $t$ sec. In what direction does the graph of T against $t$ slope ? What is the sign of $\frac{dT}{dt}$ ?

**11.** The lift $x$ in. of a cam follower after $t$ sec. is given by the following table:

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $240t$ .. | 0 to 8 | 9 | 10 | 11 | 11·2 | 11·6 | 12·0 | 12·4 | 12·8 | 13 | 14 | 15 | 16 to 24 |
| $x$ .. | 0 | 0·1 | 0·25 | 0·66 | 0·76 | 0·95 | 1·0 | 0·95 | 0·76 | 0·66 | 0·25 | 0·1 | 0 |

Draw a space-time graph for the motion of the follower, and from it draw velocity-time and acceleration-time graphs. At what values of $t$ is the velocity numerically greatest ? Describe the way in which the acceleration varies as $t$ increases from 0 to 1/10.

**12.** The following table gives the length of the day at intervals of 28 days from Jan. 1. The unit for the time $t$ is 28 days, and $l$ is the length of the day.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ .. | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $l$ .. | 7 h. 51 m. | 8 h. 54 m. | 10 h. 35 m. | 12 h. 26 m. | 14 h. 14 m. | 15 h. 46 m. | 16 h. 23 m. |

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ .. | 7 | 8 | 9 | 10 | 11 | 12 |
| $l$ .. | 16 h. 7 m. | 14 h. 35 m. | 12 h. 57 m. | 11 hr. 11 m. | 9 h. 26 m. | 8 h. 6 m. |

Draw a graph of $l$ against $t$, and from it draw a graph of $\frac{dl}{dt}$ against $t$. At what time during the year is the length of day (*a*) increasing most quickly, (*b*) decreasing most quickly ?

**13.** If a body is thrown up vertically at 80 ft./sec. its height $h$ ft. after $t$ sec. is given by $h=80t-16t^2$. Draw the graph of $h$ against $t$ from $t=0$ to 5. Find the gradients of the tangents at

12

$t=0, \frac{1}{2}, 1$, etc., and hence draw a graph of $\frac{dh}{dt}$ against $t$. At what time is $\frac{dh}{dt}$ equal to (*a*) 2 (*b*) $-2$? What is the meaning of $\frac{dh}{dt}$ being negative?

**Rate of increase with respect to other quantities than time.**

Suppose that $y$ is a function of $x$, that is, $y$ is a quantity whose value depends on the value of $x$. Let $y$ increase by $\delta y$ when $x$ increases by $\delta x$. Then the ratio of the increase in

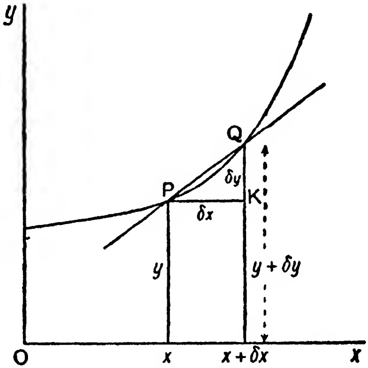



FIG. 299.

$y$ to the increase in $x$ is $\frac{\delta y}{\delta x}$. This ratio is similar to the difference ratio $\frac{\delta V}{\delta t}$ on p. 348. V and $t$ in Fig. 297 are replaced by $y$ and $x$ in Fig. 299. $\frac{\delta y}{\delta x}$ is the gradient of the chord PQ in Fig. 299, just as $\frac{\delta V}{\delta t}$ is the gradient of PQ in Fig. 297.

Hence as $\delta x$ is made smaller and smaller, $\frac{\delta y}{\delta x}$ approaches a value equal to the gradient of the tangent at P, and this is

denoted by $\frac{dy}{dx}$. It is "*the rate of increase of $y$ with respect to $x$*," just as $\frac{dV}{dt}$ is the rate of increase of V with respect to the time $t$. It is also called "*the differential coefficient of* y *with respect to* x" or "*the derivative of* y *with respect to* x."

Notice that the words "rate of increase" are used although the variable $x$ is not time. The words "with respect to $x$" show that the rate is not a time rate.

The following are examples of such rates of increase:

If a quantity of heat, Q calories, is required to raise the temperature of one gram of a liquid from some given temperature to T° centigrade, the rate of increase of Q with respect to T is the specific heat $s$ of the liquid at the temperature T°, or in symbols $s = \frac{dQ}{dT}$.

If a beam rests on supports at the same level the shearing force S at a distance $x$ from one end is the rate of increase of the bending moment M with respect to $x$, or $S = \frac{dM}{dx}$.

**Approximate method of finding a rate of increase from a table of values**

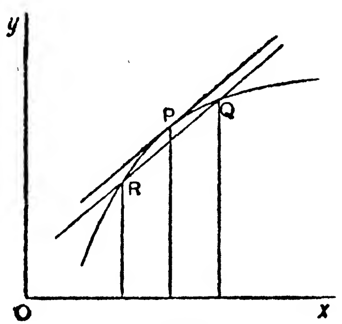



FIG. 300.

If R and Q (Fig. 300) are points near to a point P on a graph, such that the ordinate of P is midway between the ordinates at R and Q, then, provided the curve does not bend sharply between R and Q, the tangent at P is very nearly parallel to RQ. In other words the rate of increase of $y$ at P is nearly the average rate of increase from R to Q.

This method of approximating to the rate of increase of $y$ with respect to $x$ often enables us to find it directly from a table of values of $y$ and $x$, without drawing a graph, provided that the table gives values at sufficiently small intervals.

*Example.*—The following table, taken from steam tables, shows the volume V cu. ft. of 1 lb. of saturated steam at a pressure of 100 lb. per sq. in. for values of the temperature T° C. from $T=190$ to 260. From the table find the value of $\frac{dV}{dT}$, the rate of increase of V with respect to T, at $T=200$, and also tabulate the values of $\frac{dV}{dT}$ at $T=195, 205, 215$, etc. [In this case the coefficient of expansion of steam at constant pressure is given by $\frac{dV}{dT}$].

| | | | | | |
|---|---|---|---|---|---|
| T .. .. | 190 | 200 | 210 | 220 | 230 |
| V .. .. | 4·7690 | 4·8901 | 5·0101 | 5·1289 | 5·2467 |

The value of $\frac{dV}{dT}$ at $T=200$ is nearly equal to the mean rate of increase of V from $T=190$ to $T=210$.

$$\therefore \text{ at } T=200, \ \frac{dV}{dT} \doteqdot \frac{5{\cdot}0101 - 4{\cdot}7690}{20}$$

$$\doteqdot \frac{0{\cdot}2411}{20}$$

$$=0{\cdot}01205.$$

Thus the coefficient of expansion of the steam at 200° C. is 0·02105 cu. ft. per degree approximately.

The work of calculating $\frac{dV}{dT}$ at $T=195$, 205, etc., which are midway between the values of T given in the table is best set out as below:

| *T.* | *V.* | *Increase of V in 10°.* | $\frac{dV}{dT}$. |
|---|---|---|---|
| 190 | 4·7690 | | |
| 195 | | 0·1211 | 0·01211 |
| 200 | 4·8901 | | |
| 205 | | 0·1200 | 0·01200 |
| 210 | 5·0101 | | |
| 215 | | 0·1188 | 0·01188 |
| 220 | 5·1289 | | |
| 225 | | 0·1178 | 0·01178 |
| 230 | 5·2467 | | |

In this table the number 0·1211 in the third column is the increase in V, namely (4·8901 − 4·7690), in the 10° from $T=190$ to $T=200$. The number 0·01211 in the last column is obtained by dividing 0·1211 by 10, the number of degrees. This gives the average rate of increase from $T=190$ to 200, which we take to be the value of $\frac{dV}{dT}$ at $T=195$.

## Exercise XLV

1. Draw a graph of $y$ from the following table, and hence draw a graph of $\frac{dy}{dx}$ against $x$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | .. | 1 | 1·5 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $y$ | .. | 10 | 7·5 | 6·5 | 6 | 6·25 | 6·8 | 7·5 | 8·43 | 9·25 |

Find approximately the values of $x$ at which (*a*) $\frac{dy}{dx}=-\frac{2}{3}$, (*b*) $\frac{dy}{dx}=0{\cdot}5$.

2. Draw a graph of $y=x^2$ from $x=0$ to 4. Measure the gradients of the tangents at $x=0, \frac{1}{2}, 1 \ldots$ and hence plot the values of $\frac{dy}{dx}$ against $x$. Verify that the plotted points lie on a

straight line through (0, 0). Draw this straight line and find the expression of which it is the graph.

3. The following table gives the distance $y$ in. moved by a piston of a motor car engine while the crank rotates through $\theta$ degrees. Draw a graph of $y$ against $\theta$ and find from it the values of $\frac{dy}{d\theta}$ at $\theta=30$, 90 and 120 respectively.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ .. | 0 | 15 | 30 | 45 | 60 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 |
| $y$ .. | 0·77 | 0·83 | 1·02 | 1·30 | 1·66 | 2·07 | 2·46 | 2·83 | 3·16 | 3·43 | 3·62 | 3·74 | 3·77 |

4. The weight W lb. of a cubic foot of distilled water varies with the temperature T° C. according to the following table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| T .. .. | 0 | 4 | 10 | 20 | 40 | 60 | 80 |
| W .. .. | 62·34 | 62·35 | 62·33 | 62·23 | 61·87 | 61·31 | 60·89 |

Draw a graph of W against T and hence draw a graph of $\frac{dW}{dT}$ against T.

5. Plot a graph of the formula for the volume of a sphere, $V=\frac{4}{3}\pi r^3$, from $r=0$ to 2, taking values of $r$ at intervals of $\frac{1}{4}$. From the graph find the values of $\frac{dV}{dr}$ at $r=1$ and $r=1·5$.

6. From a table of logarithms to base 10, log $2·1=0·3222$, log $1·9=0·2788$. Find from these the approximate value of the rate of increase of $\log_{10} x$ with respect to $x$ when $x=2$.

7. An alternating current $i$ ampères is given by $i=10 \sin 200t$, where the angle is in radians. Calculate the values of $i$ at $t=0·0048$ and $t=0·0052$, and deduce the approximate value of $\frac{di}{dt}$ at $t=0·005$.

8. From a table of tangents find the values of tan 44° and tan 46°, and deduce the approximate value of $\frac{d(\tan x°)}{dx}$ at $x=45$.

9. Use the formula $a^2-b^2=(a+b)(a-b)$ to factorize $2·01^2-1·99^2$ and deduce the approximate value of $\frac{d(x^2)}{dx}$ at $x=2$. In the same way find the values of $\frac{d(x^2)}{dx}$ at $x=3$, 4, 5 and 6, and compare the values obtained with the values of $2x$.

10. From a table of sines of angles in radians find the value of $\frac{d(\sin x \text{ radians})}{dx}$ at $x=0$, 0·2, 0·4, 0·6, 0·8, 1·0, 1·2, 1·4, and verify

that the values obtained are very nearly equal to the corresponding values of $\cos x$.

**11.** From a table of the values of $\log_e x$ find the values of $\frac{d(\log_e x)}{dx}$ at $x=1, 2, 3, 4$ and $5$ and verify that the values obtained are nearly equal to the corresponding values of $\frac{1}{x}$.

### To find the rate of increase of *y* with respect to *x* when *y* is a given function of *x*

We have seen on p. 343 that if $V=at+b$, the rate of increase of V with respect to $t$ is $a$, or $\frac{dV}{dt}=a$. In the same way, if $y=ax+b$, $\frac{dy}{dx}=a$. We will now consider how to find the rates of increase of some other simple functions.

### Rate of increase of *y* when $y=x^2$

Fig. 301 shows the graph of $y=x^2$. At P, $x=2$, and at Q, $x=2+h$, where $h$ is any number.

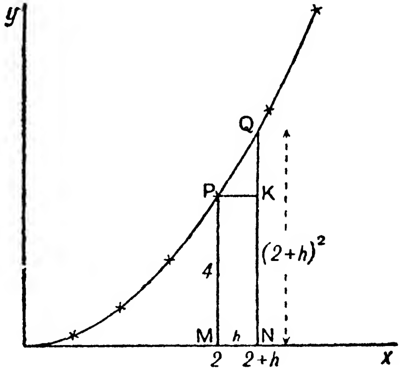



Fig. 301.

$$\therefore \text{MP} = 2^2 \text{ and NQ} = (2+h)^2$$

$\therefore$ increase in $y = \text{KQ} = (2+h)^2 - 2^2 = 4 + 4h + h^2 - 4 = 4h + h^2$,

and the increase in $x$ is $h$,

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{\text{KQ}}{\text{PK}} = \frac{4h+h^2}{h}.$$

Now whatever value $h$ has, except zero,*

$$\frac{4h+h^2}{h} = 4 + h.$$

Hence by taking $h$ smaller and smaller the ratio of the increments becomes nearer and nearer to 4. Now the value of $\frac{dy}{dx}$ at $x=2$ is the number which the difference ratio approaches as Q approaches P, that is, as $h$ approaches 0. Therefore, at $x=2$, $\frac{dy}{dx}=4$. This is the exact value of the rate of increase at $x=2$, not an approximate one, as in the case of a rate obtained graphically.

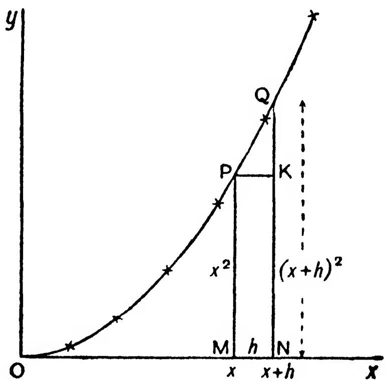



Fig. 302.

* If we put $h=0$ in $\frac{4h+h^2}{h}$ we get $\frac{0}{0}$, which has no meaning.

Now put $x$ instead of 2, and consequently $(x+h)$ instead of $(2+h)$ as in Fig. 302. Then

$$\text{increase in } y=(x+h)^2-x^2=x^2+2hx+h^2-x^2=2hx+h^2.$$

$$\therefore \frac{\text{increase in } y}{\text{increase in } x}=\frac{2hx+h^2}{h}=2x+h,$$

provided $h$ is not zero.

Now, in the same way as when $x$ was 2, the difference ratio can be made as near to $2x$ as we like by making $h$ small enough, and so the rate of increase of $y$ at the value $x$ is $2x$.

$$\therefore \text{ if } y=x^2, \frac{dy}{dx}=2x \text{ for every value of } x.$$

In obtaining $\frac{dy}{dx}$ we have used $h$ for the increment in $x$, instead of $\delta x$, because the work is easier to follow. We now write out the same equations, using $\delta x$ instead of $h$.

$$\delta y=(x+\delta x)^2-x^2=x^2+2x\,.\,\delta x+(\delta x)^2-x^2=2x\,.\,\delta x+(\delta x)^2.$$

and $$\frac{\delta y}{\delta x}=\frac{2x\,.\,\delta x+(\delta x)^2}{\delta x}=2x+\delta x.$$

As $\delta x$ is made smaller and smaller $\frac{\delta y}{\delta x}$ becomes closer and closer to $2x$,

$$\therefore \frac{dy}{dx}=2x.$$

If $y=ax^2$, where $a$ is a fixed number,

$$\delta y=a(x+\delta x)^2-ax^2=a\{(x+\delta x)^2-x^2\}.$$

We have seen above that

$$(x+\delta x)^2-x^2=2x\,.\,\delta x+(\delta x)^2$$

$$\therefore \frac{\delta y}{\delta x}=\frac{a\{2x\,.\,\delta x+(\delta x)^2\}}{\delta x}=a\{2x+\delta x\}.$$

Making $\delta x$ smaller and smaller the difference ratio approaches $a \times 2x$,

$$\therefore \frac{dy}{dx} = a \times 2x = 2ax.$$

Notice here that the factor $a$ can be written outside at each step, so that the rate of increase of $ax^2$ with respect to $x$ is $a$ times the rate of increase of $x^2$. Just as $ax^2$ is $a$ times $x^2$, so $ax^2$ increases $a$ times as fast as $x^2$.

*Example.*—The area A of a circle of radius $r$ is given by $A = \pi r^2$. Find $\frac{dA}{dr}$.

In the formula $A = \pi r^2$, A, $r$ and $\pi$ take the place of $y$, $x$ and $a$ in $y = ax^2$,

$$\therefore \frac{dA}{dr} = 2\pi r.$$

If the radius of a circle increases from $r$ to $(r + \delta r)$ as in Fig. 303 the area increases by a circular strip of width $\delta r$, and the area of this strip is the difference between the areas of the two circles which is $\pi(r + \delta r)^2 - \pi r^2$.

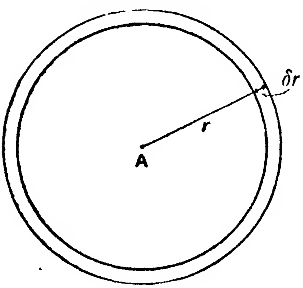



Fig. 303.

$$\therefore \delta A = \pi\{r^2 + 2r \cdot \delta r + (\delta r)^2\} - \pi r^2$$
$$= \pi r^2 + 2\pi r \cdot \delta r + \pi(\delta r)^2 - \pi r^2$$
$$\therefore \delta A = 2\pi r \cdot \delta r + \pi(\delta r)^2$$
$$\therefore \frac{\delta A}{\delta r} = 2\pi r + \pi \delta r.$$

This approaches $2\pi r$ as $\delta r$ approaches 0, and hence $\frac{dA}{dr} = 2\pi r$.

If $\delta r$ is small in the equation $\delta A = 2\pi r \cdot \delta r + \pi(\delta r)^2$, the last term is small compared with $2\pi r \cdot \delta r$, so that $\delta A \simeq 2\pi r \cdot \delta r$. The geometrical significance of this is that $2\pi r \cdot \delta r$ is the area

the strip would have if it were a *straight* strip of width $\delta r$ and length $2\pi r$., which is the circumference of the inner circle.

**The differential coefficient of $\frac{1}{x}$**

Fig. 304 shows the graph of $y=\frac{1}{x}$. P and Q are the points whose abscissæ are $x$ and $x+h$. From P to Q

$$\text{increase in } y = \frac{1}{x+h} - \frac{1}{x}$$

$$= \frac{x-(x+h)}{(x+h)x}$$

$$= \frac{-h}{(x+h)x}$$

$$\therefore \frac{\text{increase in } y}{\text{increase in } x} = \frac{\frac{-h}{(x+h)x}}{h}$$

$$= -\frac{1}{(x+h)x}.$$

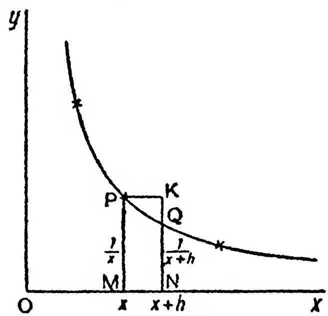



FIG. 304.

This difference ratio approaches the value $-\frac{1}{x^2}$ as $h$ approaches 0, and the value which it approaches is the value of $\frac{dy}{dx}$ at P. Hence, $\frac{dy}{dx} = -\frac{1}{x^2}$ at P.

$$\therefore \text{ when } y = \frac{1}{x}, \quad \frac{dy}{dx} = -\frac{1}{x^2}.$$

Note that actually $y$ decreases from P to Q. This is shown by a negative increase.

**The differential coefficient of $x^n$**

The rates of increase of $ax^2$ and $\frac{1}{x}$ found above are special cases of the general formula :

$$\frac{d(ax^n)}{dx} = nax^{n-1}.$$

This formula is true for all numerical values of $a$ and $n$; a proof will be given in Part III.

By putting $n=2$ in the formula we get:

$$\frac{d(ax^2)}{dx}=2ax^{2-1}=2ax.$$

In the same way, since $\frac{1}{x}=x^{-1}$,

$$\frac{d\left(\frac{1}{x}\right)}{dx}=\frac{d(x^{-1})}{dx}=-x^{-1-1}=-x^{-2}=-\frac{1}{x^2}.$$

These two results have been proved above.

*Examples.*—Using the formula for $\frac{d(ax^n)}{dx}$ find the values of

$$\frac{d(4x^{10})}{dx},\quad \frac{d\left(\frac{2}{x^3}\right)}{dx},\quad \frac{d(m^{5\cdot 8})}{dm},\quad \frac{d(4\sqrt{t})}{dt}.$$

$$\frac{d(4x^{10})}{dx}=10.4x^{10-1}=40x^9.$$

$$\frac{d\left(\frac{2}{x^3}\right)}{dx}=\frac{d(2x^{-3})}{dx}=-3.2x^{-3-1}=-6x^{-4}=-\frac{6}{x^4}.$$

$$\frac{d(m^{5\cdot 8})}{dm}=5\cdot 8m^{5\cdot 8-1}=5\cdot 8m^{4\cdot 8}.$$

$$\frac{d(4\sqrt{t})}{dt}=\frac{d(4t^{\frac{1}{2}})}{dt}=\frac{4}{2}t^{\frac{1}{2}-1}=2t^{-\frac{1}{2}}=\frac{2}{\sqrt{t}}.$$

*Example.*—When a certain gas expands adiabatically the pressure and volume are related by the equation $pv^{1\cdot 4}=C$, where C is a constant. Find $\frac{dp}{dv}$ and show that it equals $\frac{-1\cdot 4p}{v}$.

Dividing the given equation by $v^{1\cdot4}$, $p=Cv^{-1\cdot4}$,

$$\therefore\ \frac{dp}{dv}=-1\cdot4Cv^{-1\cdot4-1}=-1\cdot4Cv^{-2\cdot4}=\frac{-1\cdot4C}{v^{2\cdot4}}.$$

But $\qquad C=pv^{1\cdot4}$,

$$\therefore\ \frac{dp}{dv}=\frac{-1\cdot4pv^{1\cdot4}}{v^{2\cdot4}}=\frac{-1\cdot4p}{v^{2\cdot4-1\cdot4}}=\frac{-1\cdot4p}{v}.$$

*Example.*—The candle-power C of a certain electric lamp is related to the voltage V by the equation $C=2\cdot5\times10^{-7}V^4$. Find the differential coefficient of C with respect to V and calculate its value when (*a*) $V=200$, (*b*) $C=50$.

$$\text{Since } C=2\cdot5\times10^{-7}V^4$$

$$\therefore\ \frac{dC}{dV}=2\cdot5\times10^{-7}\,.\,4V^3=10^{-6}\,.\,V^3,$$

or the candle-power is increasing at $10^{-6}V^3$ per volt.

When $V=200$, $\dfrac{dC}{dV}=10^{-6}\times200^3=8$.

When $C=50$, $2\cdot5\times10^{-7}V^4=50$

$\therefore\ V^4=2\times10^8$

$\therefore\ V=\sqrt[4]{2}\times10^2$

$\simeq1\cdot189\times10^2=118\cdot9$.

Hence $\dfrac{dC}{dV}=10^{-6}\ V^3=\dfrac{10^{-6}V^4}{V}=\dfrac{200}{V}\simeq1\cdot682$.

**Rate of increase of the sum of two expressions**

If $y=2x^2+5x^3-3x^4$, when $x$ increases by any amount the increase in $y$

$$=\text{increase in } 2x^2+\text{increase in } 5x^3-\text{increase in } 3x^4.$$

$$\therefore\ \frac{\text{increase in } y}{\text{increase in } x}=\frac{\text{increase in } 2x^2}{\text{increase in } x}+\frac{\text{increase in } 5x^3}{\text{increase in } x}-\frac{\text{increase in } 3x^4}{\text{increase in } x}.$$

This is true however small the increase in $x$ may be, and so it must be true when each difference ratio is replaced by the rate of increase at $x$.

$$\therefore \frac{dy}{dx} = \frac{d(2x^2)}{dx} + \frac{d(5x^3)}{dx} - \frac{d(3x^4)}{dx}$$

$$= 2.2x + 5.3x^2 - 3.4x^3$$

$$= 4x + 15x^2 - 12x^3.$$

We conclude that *the rate of increase of the sum of two or more expressions is the sum of their rates of increase taken separately.*

*Example.*—A body moves $s$ ft. in $t$ sec. where $s = 12t - t^3$. Find its velocity at $t = 0, 1, 2, 3$.

$$\text{Velocity} = \dot{s} = \frac{d(12t)}{dt} - \frac{d(t^3)}{dt} = 12 - 3t^2.$$

Therefore the values of $\dot{s}$ are given by

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\dot{s}$ | 12 | 9 | 0 | −15 |

The last value shows that at $t = 3$ the body is moving back towards its starting point.

## Exercise XLVI

1. If a body fall $s$ ft. in $t$ sec. from rest, $s = 16t^2$.

   (*a*) Find the average rate at which it falls in the first 3 sec. and also the average rate at which it falls in the next second.

   (*b*) Find the average velocity from $t = 3$ to $t = 3{\cdot}1$, and from $t = 3$ to $t = 3{\cdot}01$.

   (*c*) Show that the average velocity from $t = 3$ to $t = 3 + h$ is $(96 + 16h)$ ft./sec., and from this formula find the average velocity from $t = 3$ to $t = 3{\cdot}0001$ and from $t = 3$ to $3{\cdot}000001$.

2. Show that, using the same formula $s = 16t^2$ as in Question 1, the average velocity while the time increases from $t$ to $(t+h)$ is $(32t + 16h)$. Find the average velocities for $h = 0{\cdot}01$ and $h = 0{\cdot}0001$. What is the velocity after $t$ sec.?

3. If P and Q are the points with abscissæ $x$ and $(x+h)$ on the graph of $y = 6x^2$, find the gradient of PQ and the gradient of the tangent at P.

4. If $y = 6x^2$, what is the value of $\delta y$ when $x$ increases by $\delta x$? Find the value of $\frac{\delta y}{\delta x}$ and deduce from it the value of $\frac{dy}{dx}$.

5. If $y = \frac{1}{x}$ find the value of $\delta y$ when $x$ increases by $\delta x$. Hence find the value of $\frac{dy}{dx}$.

6. If $y = x^3$ find the increase in $y$ when $x$ increases by $h$, and hence find the ratio of the increase in $y$ to the increase in $x$ in its simplest form. Deduce the value of $\frac{dy}{dx}$.

7. If $y = ax + b$ show that $\frac{\delta y}{\delta x}$ has the same value for all values of $\delta x$. What is the meaning of this? What is the value of $\frac{dy}{dx}$?

8. Find the rate of increase of $\frac{1}{x^2}$ with respect to $x$ by the same method as in question 5.

In the following questions use the formula $\frac{d(x^n)}{dx} = nx^{n-1}$.

Find the values of $\frac{dy}{dx}$ when $y$ is:

9. $3x^3$. 10. $4 - 2x^2$. 11. $90x^8$. 12. $2x^4 - \frac{1}{3}x^6$.

13. $4x^3 - 7x^2 + 21x - 50$. 14. $0{\cdot}02x - 2{\cdot}54x^2 - 0{\cdot}72x^3$.

15. $\frac{1}{x^3}$. 16. $\frac{1}{8x^4}$. 17. $3x^{1{\cdot}6}$. 18. $\frac{1}{4x^2} - \frac{1}{2x}$.

Find the values of the following differential coefficients:

19. $\frac{d(5t^2)}{dt}$. 20. $\frac{d(\frac{1}{5}n^3)}{dn}$. 21. $\frac{d(0{\cdot}1r^2)}{dr}$.

22. $\dfrac{d(2000x^4)}{dx}$. 23. $\dfrac{d(7l^7)}{dl}$. 24. $\dfrac{d(4p^2+5p^4)}{dp}$.

25. $\dfrac{d\left(\frac{1}{n}x^n\right)}{dx}$. 26. $\dfrac{d(ut+\frac{1}{2}gt^2)}{dt}$. 27. $\dfrac{d\left(\frac{100}{v}\right)}{dv}$.

28. $\dfrac{d\left(\frac{1}{r^k}\right)}{dr}$. 29. $\dfrac{d(2\times 10^{-6}\times V^{4\cdot 8})}{dV}$. 30. $\dfrac{d(4\sqrt{z})}{dz}$.

31. Find the gradient of the tangent to the curve $y=x^2-x$ at $x=\frac{1}{4}$ and at $x=1\frac{1}{4}$. For what value of $x$ is the gradient of the tangent zero? Find the value of $y$ at this point.

32. If a body is thrown upwards at 80 ft./sec. under gravity its height $h$ ft. after $t$ sec. is given by $h=80t-16t^2$. Find its velocity at $t=2$ and $t=3$. When is its velocity zero and how high is it then?

33. Find the gradients of the tangents to $y=x^2-3x+2$ at the points where the curve crosses the axis of $x$.

34. Show that the rate of increase of the volume of a sphere with respect to the radius equals the area of the surface of the sphere.

35. Find the co-ordinates of the points on the graph of $y=x^3-3x^2+6$ at which the tangents are horizontal.

## Maxima and minima

To find for what values of $x$ the value of $y$ is a maximum or a minimum we have to find the values of $x$ at which the tangent to the graph is horizontal (as at A and B in Fig. 303); these are the values of $x$ which make $\dfrac{dy}{dx}=0$. If the curve is below the tangent on both sides as at A, the value of $y$ is a maximum at A, whereas, if the curve is above the tangent as at B, the value of $y$ is a minimum.

Now just to the left of A the value of $\dfrac{dy}{dx}$ is positive, and just to the right of A it is negative.

Therefore $y$ has a maximum value at A if $\frac{dy}{dx}=0$ at A, and if $\frac{dy}{dx}$ is positive just to the left of A, and negative just to the right of A.

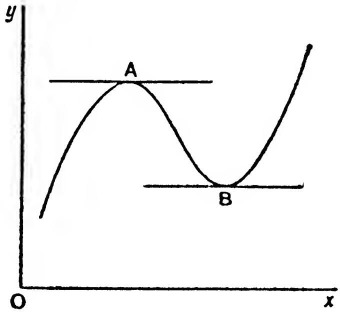



FIG. 305.

In the same way it can be seen that $y$ has a minimum value at B if $\frac{dy}{dx}=0$ at B, and if $\frac{dy}{dx}$ is negative just to the left of B, and positive just to the right of B.

*Example.*—Find the greatest rectangular area that can be enclosed by 400 yds. of fencing.

If one side of the rectangle is $x$ yds. the length of a perpendicular side is $(200-x)$ yds. Hence the area A sq. yds. is given by :

$$A=x(200-x)=200x-x^2$$

$$\frac{dA}{dx}=200-2x$$

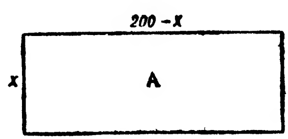



FIG. 306.

$$\therefore \frac{dA}{dx}=0 \text{ when } 200-2x=0,$$

from which $x=100$.

When $x=100$, $A=100\times 100=10{,}000$.

If $x$ is just less than 100, say 99·9, $\frac{dA}{dx}$ is positive, and if $x$ is just greater than 100, $\frac{dA}{dx}$ is negative. Therefore, 10,000 is the maximum value of A.

Fig. 307 shows the graph of A against $x$.

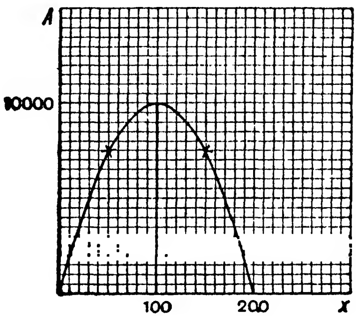



Fig. 307.

## Exercise XLVII

Find any maximum and minimum values of $y$, and sketch a rough graph of $y$ when :

1. $y=2x-x^2$.  2. $y=x^3-3x^2$.  3. $y=1-x-x^2$.

4. $y=t^2-6t$.  5. $y=1-2v+v^2$.  6. $y=r^3-3r$.

7. The bending moment of a certain beam at a distance $x$ from one end is given by $M=\frac{1}{2}wx(l-x)$. Find its maximum value.

8. A closed tank is to be made with a square base to hold 40 gallons. If the edge of the tank is $x$ ft., show that its surface area S ft.$^2$ is given by $S=2x^2+\frac{25\cdot6}{x}$. Hence find its most economical shape. [1 gallon of water occupies 0·16 cu. ft.]

9. The horse-power transmitted by a belt moving at $v$ ft./sec. is proportional to $fv-\frac{wv^3}{g}$, where $f$ lb. wt./ft.$^2$ is the maximum allowable stress in the belt, and $w$ lb./ft.$^3$ is the weight per unit

volume of the belt. If $f=23{,}800$ and $w=62$, find the value of $v$ for which the horse-power is a maximum.

**10.** The power W watts given to an external circuit by a battery of internal resistance $r$ and electromotive force E volts when the current is $i$ ampères, is given by $W=Ei-ri^2$. If $E=12$, $r=16$, find for what current W is a maximum.

**11.** If $y=4x-x^2$, find the least and greatest values of $y$ between $x=1$ and $x=2{\cdot}5$.

**12.** If a beam AB of weight W lb. wt. and length $l$ ft. is clamped horizontally at B and supported at A at the same level, its bending moment M lb. wt. ft. at $x$ ft. from A is given by $M=\frac{W}{l}(\frac{3}{8}lx-\frac{1}{2}x^2)$. Show that M has a maximum value at $x=\frac{3}{8}l$, and calculate this maximum. Sketch a graph of M and use it to show that M is *numerically* greatest at $x=l$.

# CHAPTER XVI

# INTEGRATION

## To find the distance moved by a body from its velocity-time graph

If the velocity $v$ is uniform, the distance it moves in a time $t$ is $vt$.

The graph of the velocity against the time is the straight line DC in Fig. 308. In this figure OB represents $t$, OD represents $v$ and hence the distance moved $vt$ is represented by OD × OB, which is the area of the rectangle OBCD. For convenience we shall say that the distance moved is equal to the area OBCD.

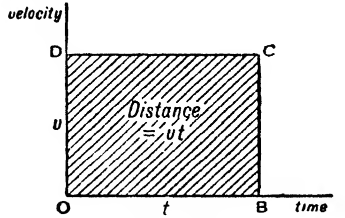


FIG. 308.

**Variable velocity**

When a body is moving with variable velocity its velocity-time graph is a curve, such as that in Fig. 309. Suppose we require to find the distance it moves in the time AB from $t=a$ to $t=b$. Divide the area ABCD into a number of strips by ordinates at equal intervals of $t$. During one of these intervals, such as MN, the velocity is nearly uniform and its value is nearly equal to PM throughout the interval. Hence

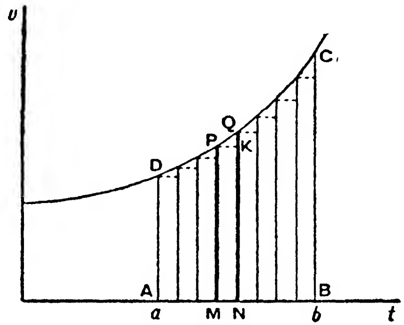



FIG. 309.

the distance the body moves in the time MN is nearly equal to the area PMNK. Similarly, the distance moved in each interval is nearly equal to the area under the dotted horizontal line, and therefore, adding up the distances moved in the intervals, the whole distance moved in the time AB is nearly equal to the total area under the dotted lines. By dividing AB into smaller and smaller intervals of time the total area under the dotted lines approximates more and more closely to the distance moved and also to the area ABCD bounded by the $t$ axis, the curve DC and the ordinates AD and BC, which is called the area under the graph from $t=a$ to $t=b$. For this reason we conclude that the distance travelled in the time from $t=a$ to $t=b$ is *exactly* equal to the area ABCD under the graph of the velocity $v$ from $t=a$ to $t=b$.

The following example shows how the distance can be calculated.

*Example.*—During the minute after the instant at which an accelerating car has attained a velocity of 12 m.p.h. its velocity is given by :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Time (sec.) .. | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
| Velocity (m.p.h.) | 12 | 34 | 49 | 59 | $65\frac{1}{2}$ | 69 | 70 |

Find the distance it moves in (*a*) the whole minute, (*b*) the last 20 sec.

VELOCITY-TIME GRAPH FOR MOTION OF A CAR

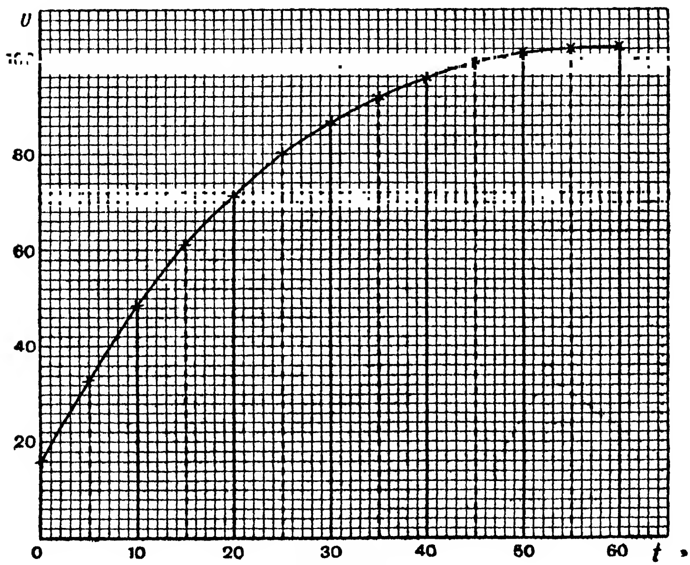



Fig. 310.

The work is simplified by first finding the velocity in ft. per sec.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ (sec.) .. | 0 | 10 | 20 | 30 | 40 | 50 | 60 |
| $v$ (ft./sec.) .. | 17·6 | 49·9 | 71·9 | 86·5 | 96·1 | 101·2 | 102·7 |

From this table a graph is drawn in Fig. 310. The area under it is divided into six strips by ordinates at intervals of 10 sec. The ordinates at the middle of each interval are shown by dotted lines. The area of each strip is nearly equal to mid-ordinate × base.

$\therefore$ Distance travelled in 1 min.

$$
\begin{aligned}
&= \text{area under graph from } t=0 \text{ to } t=60\\
&\simeq 10\{\text{sum of mid-ordinates}\}\\
&\simeq 10\{33{\cdot}4+61+80+91{\cdot}6+98{\cdot}4+102{\cdot}2\}\\
&\simeq 10\times 466{\cdot}6 \text{ ft.}\\
&\simeq 4670 \text{ ft.}
\end{aligned}
$$

Also, distance moved in last 20 sec.

$$
\begin{aligned}
&\simeq 10\{98{\cdot}4+102{\cdot}2\}\\
&\simeq 2010 \text{ ft.}
\end{aligned}
$$

*Example.*—A mine cage drops for a time at uniform acceleration and then slows up with uniform retardation. If it comes to rest after it has dropped 800 ft. in 25 sec. find its maximum speed.

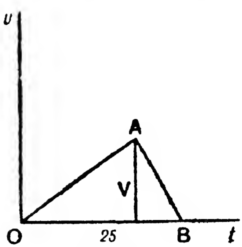



FIG. 311.

Since the cage starts from rest at uniform acceleration the velocity-time graph is a straight line OA through the origin until the cage stops accelerating. Then, while the cage is slowing up the velocity-time graph is another straight line AB because the retardation is constant.

The maximum velocity V is the ordinate of A, and the total distance travelled is the area under the graph, i.e. the area of the triangle OAB. But the area of the triangle is $\frac{1}{2}V\times 25$.

$$\therefore\ \tfrac{1}{2}V\times 25=800$$

$$\therefore\ V=64 \text{ ft./sec.}$$

## Exercise XLVIII

1. The relationship between the time and velocity of a body is given by :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ sec. .. | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $v$ ft./sec. .. | 0 | 0·5 | 3 | 7·5 | 14 | 22·5 | 35·5 |

Draw a graph of $v$ against $t$, and deduce the distance passed over in 6 sec.

2. When a train is travelling at 40 m.p.h. steam is shut off and the brakes applied. The following table gives its speed after $t$ sec.

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ sec. .. | 0 | 12 | 24 | 36 | 48 | 60 |
| $v$ m.p.h. .. | 40 | 30 | 23 | 20·4 | 14 | 12 |

How far does the train travel in this time?

3. A flywheel slowing down has speeds given by the following table :

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ min. .. .. | 0 | 1 | 2 | 3 | 4 | 5 |
| N. revs. per min. | 320 | 200 | 116 | 60 | 23 | 0 |

Draw a graph of N against $t$, and from it find the total number of turns the flywheel makes in coming to rest.

4. If the velocity-time graph for the motion of a train between two stations is OABC shown in Fig. 312, find the distance between the stations. Find also the acceleration in ft./sec.$^2$ while the train is getting up speed.

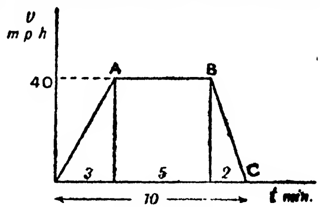


FIG. 312.

### Space-time graph from the velocity-time graph

The following table gives the velocity, $v$ miles per hour, of an electric train as it goes from one station to another at $t$ hr. after it has left the first station. The first line gives the values of $60t$, which is the time in minutes.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $60t$ .. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7—13 | 14 | 15 | 16 |
| $v$ .. | 0 | | 39 | 47 | 52 | 54 | 55·7 | 56 | 50 | 43 | 0 |

From 7th to 13th minute the velocity is constant, so that this part of the graph of $v$ is a straight line. Fig. 313 shows the graph of $v$. The area under it is divided into strips by ordinates at intervals of one minute.

VELOCITY-TIME GRAPH FOR AN ELECTRIC TRAIN

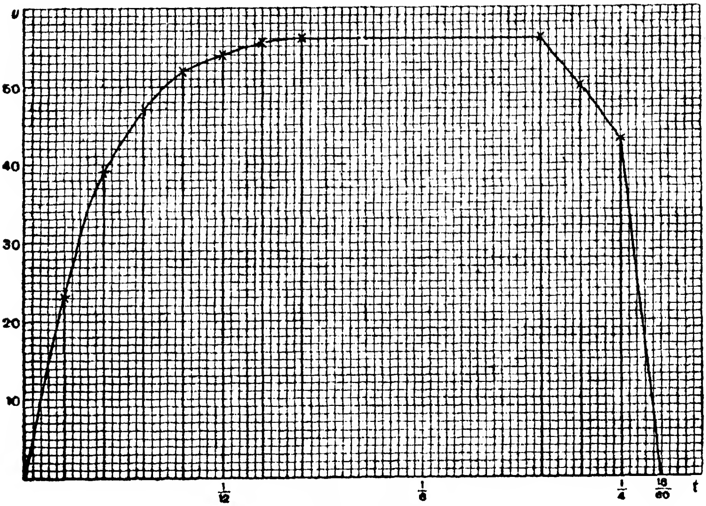



FIG. 313.

The distance in miles travelled in the first minute is the area of the first strip, i.e. it is represented by this area.

The following table gives the mid-ordinates of the strips in miles per hour, the distance travelled in each minute (except from $t=\frac{7}{60}$ to $t=\frac{13}{60}$ where the whole distance travelled in this time is given), and in the last column the distance travelled in the first $t$ hr. In this column :

the distance from $t=0$ to $t=\frac{2}{60}=0{\cdot}217+0{\cdot}533=0{\cdot}750$.

Also : $1{\cdot}473=$ the distance from $t=0$ to $t=\frac{3}{60}$

$=$ (distance from $t=0$ to $t=\frac{2}{60}$) + (distance from $t=\frac{2}{60}$ to $t=\frac{3}{60}$)

$=0{\cdot}750+0{\cdot}723=1{\cdot}473$.

| *t hr.* | *Mid-ordinate in m.p.h.* | *Distance travelled (ml.).* | *Distance in t hr.* |
|---|---|---|---|
| 0 | | | |
| | 13·0 | 0·217 | |
| $\frac{1}{60}$ | | | 0·217 |
| | 32·0 | 0·533 | |
| $\frac{2}{60}$ | | | 0·750 |
| | 43·4 | 0·723 | |
| $\frac{3}{60}$ | | | 1·473 |
| | 49·4 | 0·823 | |
| $\frac{4}{60}$ | | | 2·296 |
| | 53·0 | 0·883 | |
| $\frac{5}{60}$ | | | 3·179 |
| | 55·0 | 0·917 | |
| $\frac{6}{60}$ | | | 4·096 |
| | 55·8 | 0·930 | |
| $\frac{7}{60}$ | | | 5·026 |
| | 56·0 | 5·600 | |
| $\frac{13}{60}$ | | | 10·626 |
| | 55·0 | 0·883 | |
| $\frac{14}{60}$ | | | 11·509 |
| | 46·2 | 0·770 | |
| $\frac{15}{60}$ | | | 12·279 |
| | 21·4 | 0·357 | |
| $\frac{16}{60}$ | | | 12·636 |

From this table a graph of the distance travelled is plotted against $t$; it is the thick curve in Fig. 314. This graph can be used to find how long the train takes for any given distance; e.g. when the distance is 9 miles, $t=(11{\cdot}25/60)$, so the train takes (11·25/60) hr., i.e. $11\frac{1}{4}$ min. for the first 9 ml.

If the distance from the first station is $s$ ml., then this is the graph of $s$ against $t$, but instead of $s$ being measured from the station it might be measured from a station two miles farther back, then the graph of $s$ is the dotted curve in Fig. 314, which is 2 units above the other curve.

Wherever $s$ is measured from, $v=\dfrac{ds}{dt}$ and hence the area under the graph of $\dfrac{ds}{dt}$ for any interval of time, say from $t=a$ to $t=b$, is equal to the increase in the value of $s$ from $t=a$ to $t=b$.

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SPACE-TIME GRAPH FOR AN ELECTRIC TRAI... rease of

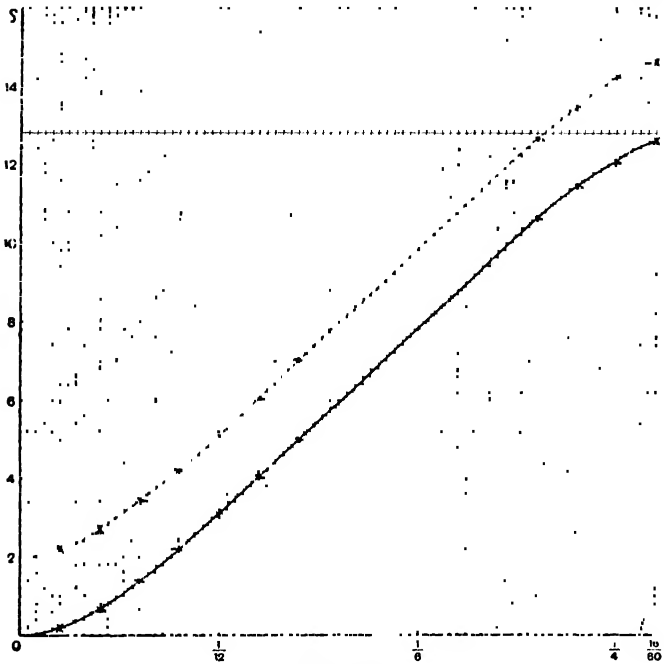


Fig. 314.

## Integration

By dividing up the area under the velocity-time graph into a large number of very small parts we have seen that the distance travelled is exactly equal to the area under the velocity-time graph. This process of calculating the value of a quantity by dividing it up into a larger and larger number of smaller and smaller parts is called integration. Just as the word " integer " means " whole number," so " integration " means finding the whole from the sum of the parts. This method was first used by Archimedes, who lived in Syracuse from 287 B.C.

is nearly .., to find the volumes of a sphere and a cone, and to under other similar problems.

## The relationship between differentiation and integration

We have seen that the area under the graph of $\frac{ds}{dt}$ for any interval of time, say from $t=a$ to $t=b$, is the distance travelled by the body in that time or the increase in the value of $s$ in that time.

Thus the graph of $\frac{ds}{dt}$ is found from the graph of $s$ by measuring the gradients of tangents, or differentiation, and the graph of $s$ can be found from the graph of $\frac{ds}{dt}$ (provided we know the value of $s$ at $t=0$) by adding up areas, or integration. The graph of $s$ is called an integral curve or sum curve of the graph of $t$.

## Integration with respect to any variable

If we write $z$ in place of $s$ and $x$ in place of $t$ the differential coefficient $\frac{ds}{dt}$ becomes $\frac{dz}{dx}$, and hence the area under the graph of $\frac{dz}{dx}$ from $x=a$ to $x=b$ equals the increase of $z$ from $x=a$ to $x=b$. We will consider below the special case of $z=x^3$.

## The area under the graph of $3x^2$ from $x=a$ to $x=b$

If $z=x^3$, $\frac{dz}{dx}=3x^2$. Fig. 315 shows the graph of $x^3$ above the graph of $3x^2$. The area under the graph of $3x^2$ from $x=a$ to $x=b$ is divided into a number of strips by ordinates at equal intervals of $x$. Ordinates are also drawn to the graph of $x^3$ at the same intervals. Consider the area of one of the

strips PMNQ under the graph $3x^2$. Since $\dfrac{d(x^3)}{dx}$ ises from $=$ ease of ordinate MP is equal to the gradient of the tangent at $p$ on the graph of $x^3$, which is approximately $\dfrac{kq}{pk}$.

$$\therefore \text{ MP} \simeq \frac{kq}{pk} = \frac{kq}{\text{MN}}.$$

$$\begin{aligned} \therefore \text{ the area of PMNK} &= \text{MP} \times \text{MN} \\ &\simeq \frac{kq}{\text{MN}} \times \text{MN} \\ &\simeq kq. \end{aligned}$$

Thus each of the steps similar to $kq$ along the graph of $x^3$

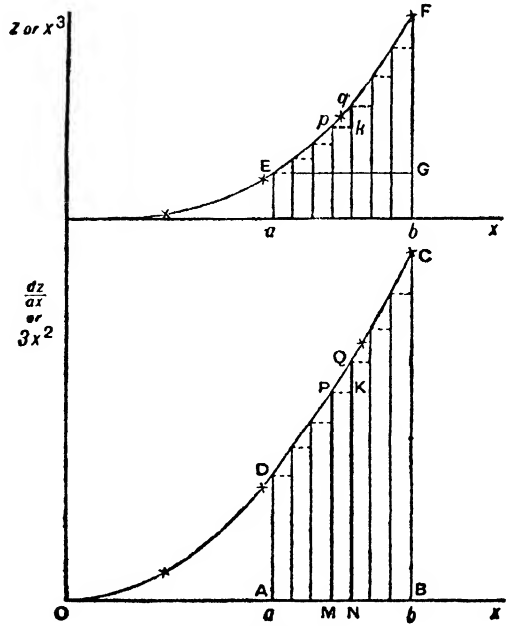


Fig. 315.

is nearly equal to the area of the corresponding rectangle under the graph of $3x^2$.

Hence, by addition, GF, the total increase in $x^3$ from E to F, is nearly equal to the sum of the areas of the rectangles below the dotted lines in the lower graph.

If the intervals are made smaller and smaller this approximation becomes more and more accurate. Hence we conclude that the area ABCD under the graph of $3x^2$ from $x=a$ to $x=b$ is exactly equal to GF, the increase of $x^3$ as $x$ increases from $a$ to $b$.

But at E, $x^3=a^3$, and at F, $x^3=b^3$.

$\therefore$ GF $=b^3-a^3$.

$\therefore$ area under the graph of $3x^2$ from $x=a$ to $x=b$ is $(b^3-a^3)$.

$\Big[x^3\Big]_a^b$ is written as short for the words "the increase of $x^3$ from $x=a$ to $x=b$." Using this notation, the area under the graph of $3x^2$ from $x=1$ to $x=2$

$$=\Big[x^3\Big]_1^2=2^3-1^3=7.$$

*Example.*—Find the area under the graph of $x^3$ from $x=2$ to $x=4$.

The differential coefficient of $x^4$ is $4x^3$.

$\therefore$ if $z=\frac{1}{4}x^4$, $\dfrac{dz}{dx}=\frac{1}{4}\,.\,4x^3=x^3$.

$\therefore$ area under the graph of $x^3$ from $x=2$ to $x=4$.

$$=\Big[\tfrac{1}{4}x^4\Big]_2^4=\tfrac{1}{4}4^4-\tfrac{1}{4}\,.\,2^4=64-4=60.$$

Every expression such as $\frac{1}{4}x^4+7$, $\frac{1}{4}x^4-0{\cdot}3$, $120+\frac{1}{4}x^4$ obtained by adding a constant to $\frac{1}{4}x^4$ has the same differential coefficient as $\frac{1}{4}x^4$, namely $x^3$. Every such expression is included in $\frac{1}{4}x^4+c$ where $c$ is any constant whatever. Thus we could write :

Area under the graph of $x^3$ from $x=2$ to $x=4$

$$=\Big[\tfrac{1}{4}x^4+c\Big]_2^4=[\tfrac{1}{4}4^4+c]-[\tfrac{1}{4}2^4+c]=64+c-4-c=60.$$

Hence the addition of $c$ makes no difference to the calculation of the area under a curve between two ordinates. It is equivalent to the addition of 2 to $s$ to get the upper graph in Fig. 314.

## Exercise XLIX

**1.** The speed $v$ ft./sec. of a motor car after $t$ sec. from rest is given by:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ .. | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
| $v$ .. | 0 | 12·6 | 19·9 | 26·7 | 33·7 | 44·4 | 50·4 |

Draw a velocity-time graph and from it find by the mid-ordinate rule the distances described in successive intervals of five seconds. Hence make a table of the distance $s$ ft. described in $t$ sec., and draw a graph of $s$ against $t$. From it estimate the time taken for the first 200 yds.

**2.** When a condenser is being charged the current flowing into the plate is given by:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $t$ sec. .. | 0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 |
| $i$ $m$A .. | 200 | 122 | 74 | 44 | 28 | 16 | 10 |

Assuming that the current in milli-amperes is the rate of increase of the charge Q milli-coulombs on the plate, and that $Q = 0$ at $t = 0$, draw a graph to show the values of Q from $t = 0$ to $t = 0{\cdot}6$.

**3.** When a voltage V is applied to an inductance of $\frac{1}{2}$ henry the voltage is the rate of increase of $\frac{1}{2}i$, where $i$ is the current in ampères, or $V = \frac{1}{2}\frac{di}{dt}$. The following table gives the values of the voltage at intervals of 0·01 sec. Given that $i = 0$ at $t = 0$, draw a graph of $i$ against $t$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $t$ .. | 0 | 0·01 | 0·02 | 0·03 | 0·04 | 0·05 |
| V .. | 200 | 198 | 162 | 118 | 62 | 0 |

From the graph find the value of $t$ when $i = 8{\cdot}5$.

**4.** Using the table in Question 1, p. 375, draw a graph of the distance described against the time for values of $t$ from 0 to 6.

**5.** From the table in Question 2, p. 375, draw a graph of the distance $s$ ft. described in $t$ sec. from $t = 0$ to $t = 60$.

**6.** During one revolution of a cam the follower is required to have an acceleration in ft./sec.$^2$ given by the following table, the

first value being when the follower is at rest in its lowest position. Draw the acceleration-time graph and deduce from it graphs to show the velocity in ft./sec. and the lift of the follower in inches at any time.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | .. | 0 | 0·01 | 0·02 | 0·03 | 0·04 | 0·05 | 0·06 | 0·07 | 0·08 | 0·09 |
| $\dot{v}$ | .. | 0 | 35 | 120 | 200 | 200 | 200 | 200 | 120 | 35 | 0 |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | .. | 0·10 | 0·11 | 0·12 | 0·13 | 0·14 | 0·15 | 0·16 | 0·17 | 0·18 |
| $\dot{v}$ | .. | −35 | −120 | −200 | −200 | −200 | −200 | −120 | −35 | 0 |

**7.** Draw a curve from the following table and deduce from it an integral curve, showing the integral of $y$ with respect to $x$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | .. | 0 | 1 | 2 | 3 | 4 | 5 |
| $y$ | .. | 6·00 | 7·33 | 9·00 | 10·80 | 12·67 | 14·57 |

**8.** If the values of $\frac{dz}{dx}$ are given by the following table, and $z=2$ at $x=0$, draw a graph of $z$. Find the value of $x$ which makes $z=5$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | .. | 0 | 2 | 4 | 6 | 8 |
| $\frac{dz}{dx}$ | .. | 0·71 | 1 | 1·25 | 1·42 | 1·58 |

**9.** If $\frac{dz}{dx}=2x$, what is the value of $z$? Find the area under the graph of $2x$ from $x=a$ to $b$. Draw the graph and find the area by using the formula for the area of a trapezium.

**10.** If $\frac{ds}{dt}=\frac{1}{2}t^2$, what is the value of $s$? Find the value of $s$ at $t=2$ if $s=1$ at $t=0$.

**11.** A body describes $s$ ft. in $t$ sec. If its velocity after $t$ sec. is $32t$, find how far it goes from $t=1$ to $t=4$.

Write down the functions whose rates of increase with respect to $x$ are:

**12.** $6x$. **13.** $6x+5$. **14.** $9x^2$.

**15.** $9x^2+6x+5$. **16.** $10x^5$. **17.** $\frac{1}{x^2}$.

**18.** $1{\cdot}8x^{0{\cdot}8}$. **19.** $6x+\frac{1}{x^2}$.

**The notation used in integration**

Let us now consider how the area under $y=3x^2$ from $x=1$ to $x=2$ is obtained by adding small strips (this is the area found on p. 381). Suppose the area is divided up into ten strips by ordinates at 1·1, 1·2, 1·3, etc., as in Fig. 316. The sum of the areas of these strips is nearly :

$$(3\times1^2)0{\cdot}1+(3\times1{\cdot}1^2)0{\cdot}1+(3\times1{\cdot}2^2)0{\cdot}1+\ .\ .\ .\ +(3\times1{\cdot}9^2)0{\cdot}1.$$

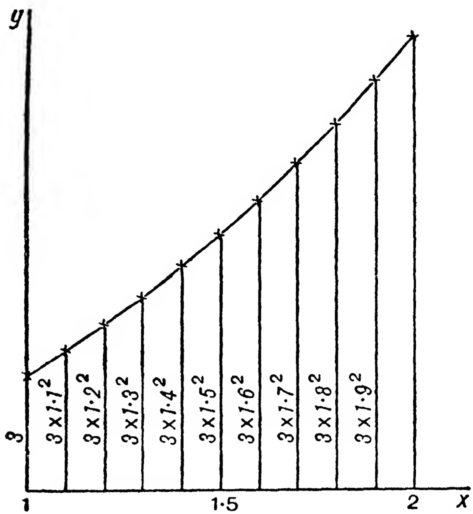

Fig. 316.

If we denote the small division 0·1 by the symbol $\delta x$ then this sum is the sum of all the values of $3x^2\,.\,\delta x$ obtained by putting $x=1,\ 1{\cdot}1,\ 1{\cdot}2,\ .\ .\ .\ 1{\cdot}9$. We use the shorthand $\sum\limits_{x=1}^{x=2} 3x^2\,.\,\delta x$ for this sum ; $\Sigma$ being the capital sigma of the Greek alphabet, and standing here for the words " the sum of all the values of."

Since $\sum\limits_{x=1}^{x=2} 3x^2\,.\,\delta x$ is only an approximation for this area, a new notation is needed for the exact area. The notation

that everyone uses to-day is the notation invented by Leibniz, $\int_1^2 3x^2dx$. In this notation $\int$ is an elongated S, standing for "sum," just as $\Sigma$ does, but with this difference, that $\int_1^2 3x^2dx$ implies that in finding the sum the elements $3x^2 \,.\, \delta x$ have been made smaller and smaller. It is read, "the integral of $3x^2dx$ from $x=1$ to $x=2$," and $x=1$ and $x=2$ are called the lower and upper limits of integration. We have seen that the area under $y=3x^2$ from $x=1$ to 2 equals the increase in the value of $x^3+c$ from $x=1$ to 2, because

$$\frac{d(x^3+c)}{dx}=3x^2.$$

Hence $$\int_1^2 3x^2dx=\Big[x^3+c\Big]_1^2=(2^3+c)-(1^3+c)=7.$$

For this reason we write :

$$\int 3x^2dx=x^3+c.$$

$\int 3x^2dx$ is called the *indefinite integral* of $3x^2dx$ because its value $(x^3+c)$ depends on $x$. It does not give the definite area until we have substituted numerical values of $x$. On the other hand $\int_1^2 3x^2dx$ is called a *definite integral* because it has a definite value, and equals the area under $y=3x^2$ from $x=1$ to $x=2$.

In the same way, since

$$\frac{d(x^4+c)}{dx}=4x^3,$$

$$\int 4x^3dx=x^4+c,$$

and since $$\frac{d(x^{n+1}+c)}{dx}=(n+1)x^n,$$

$$\int (n+1)x^ndx=x^{n+1}+c.$$

This last result suggests differentiating $\dfrac{ax^{n+1}}{n+1}$ in order to find the integral of $ax^n dx$.

$$\frac{d\left(\dfrac{ax^{n+1}}{n+1}+c\right)}{dx}=\frac{(n+1)ax^n}{n+1}=ax^n.$$

$$\therefore \int ax^n dx=\frac{ax^{n+1}}{n+1}+c.$$

---

This is the formula for integrating any power of $x$ except $x^{-1}$. [The integral of $x^{-1}$ will be found in Part III.] Using suitable values of $a$ and $n$ we find,

$$\int 2x^4 dx=\frac{2x^{4+1}}{4+1}+c=\frac{2x^5}{5}+c.$$

$$\int \tfrac{1}{3}t^7 dt=\tfrac{1}{3}\cdot\frac{t^{7+1}}{7+1}+c=\frac{t^8}{24}+c.$$

If $l$ is a constant, $\displaystyle\int\left(\frac{z}{l}\right)^3 dz=\int\frac{1}{l^3}\cdot z^3 dz=\frac{1}{l^3}\cdot\frac{z^4}{4}+c=\frac{z^4}{4l^3}+c.$

$$\int\frac{1}{x^2}dx=\int x^{-2}dx=\frac{x^{-2+1}}{-2+1}+c=-x^{-1}+c=-\frac{1}{x}+c.$$

$$\int r^{2\cdot 7}dr=\frac{r^{2\cdot 7+1}}{2\cdot 7+1}+c=\frac{r^{3\cdot 7}}{3\cdot 7}+c.$$

*Example.*—Find the area under $y=\frac{1}{10}x^3$ from $x=2$ to $x=3$.

$$\text{Area}=\int_2^3 \tfrac{1}{10}x^3 dx=\left[\tfrac{1}{10}\frac{x^4}{4}+c\right]_2^3$$

$$=\left(\tfrac{1}{10}\frac{3^4}{4}+c\right)-\left(\tfrac{1}{10}\cdot\frac{2^4}{4}+c\right)$$

$$=\frac{81-16}{40}=\tfrac{13}{8}.$$

Notice again that in finding any definite integral it is not necessary to include the constant $c$. However, in all other applications of integration except those involving the evaluation of definite integrals the constant should be included. The problem that follows illustrates this.

*Example.*—If a body has an acceleration $\frac{1}{2}t^2$ after $t$ sec., find its velocity after $t$ sec. given that its velocity at $t=0$ is 2 ft./sec.

If the velocity is $v$ ft./sec.,

$$\frac{dv}{dt}=\tfrac{1}{2}t^2.$$

$$\therefore\ v=\int \tfrac{1}{2}t^2dt$$

$$=\tfrac{1}{2}\cdot\frac{t^3}{3}+c=\tfrac{1}{6}t^3+c.$$

But $\qquad v=2$ at $t=0$,

$$\therefore\ 2=0+c.$$

$$\therefore\ c=2.$$

$$\therefore\ v=\tfrac{1}{6}t^3+2.$$

## Exercise L

Using the formula $\int ax^n dx=\dfrac{ax^{n+1}}{n+1}+c$, find :

1. $\int x^7dx.$ 2. $\int 10x^4dx.$ 3. $\int \frac{1}{2}v^4dv.$ 4. $\int 0{\cdot}1t^9dt.$

5. $\int \sqrt{r}dr.$ 6. $\int \dfrac{dx}{2x^2}.$ 7. $\int \dfrac{dm}{m^4}.$ 8. $\int 12p^{2\cdot 6}dp.$

Evaluate the following integrals :

9. $\int_1^3 2xdx.$ 10. $\int_0^2 6x^2dx.$ 11. $\int_{\frac{1}{2}}^{1\frac{1}{2}} x^3dx.$ 12. $\int_0^a 4\pi r^2dr.$

13. $\int_{\frac{1}{2}}^{1} t^4dt.$ 14. $\int_2^3 \dfrac{dx}{x^2}.$ 15. $\int_1^2 \dfrac{20dv}{t^{1\cdot 2}}.$ 16. $\int_2^4 \dfrac{dt}{3\sqrt{t}}.$

17. $\int_{-a}^{+a} x^6 dx.$ 18. $\int_{v_1}^{v_2} \frac{dv}{v^n}.$ 19. $\int_{0}^{+1} k^{2.6} dk.$

Find the values of :

20. $\int (1 - x^2) dx.$ 21. $\int (2v - 4v^3) dv.$ 22. $\int (x^3 - x) dx.$

Find the following areas :

23. Under the curve $y = 4x^2$ from $x = 0$ to 3.

24. Under the curve $y = x^n$ from $x = a$ to $b$ [$n$ not equal to $-1$].

25. Under the curve $y = 2\sqrt{x}$ from $x = 1$ to 4.

26. Find by integration the area under the line $y = x$ from $x = a$ to $b$. Verify your result by using the formula for the area of a trapezium.

27. If the velocity of a body is $32t$ ft./sec. after $t$ sec., express the distance travelled from $t = 1$ to 2 as an integral and evaluate the integral.

28. If the acceleration of a body is $4t$ ft./sec.$^2$ after $t$ sec., find its velocity $v$ ft./sec. after $t$ sec., given that $v = 6$ at $t = 0$. How fast is it travelling after 5 sec. ?

29. Draw the graph of $y = x^2$ from 0 to 2. Divide the area which is bounded by the curve, the axis of $x$ and the ordinate at $x = 2$, into ten strips by ordinates at $x = 0{\cdot}2$, $0{\cdot}4$, etc. Hence draw an integral curve. Verify that the gradients of the integral curve at $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$ are the values of $y$ at these values of $x$. Find, by using the formula for $\int x^2 dx$, the equation of the integral curve.

30. Find the co-ordinates of the points A and B in which the graph of $y = 5x - x^2 - 6$ cuts the axis of $x$. Hence find the area bounded by AB and the part of the graph which is above AB.

## Applications of integration.

*The volume of a solid*

If the cross-sections of a solid by parallel planes have the same area A and the length of the solid is $x$, the volume of the

solid is $Ax$, which is the area of the rectangle under the graph of A against $x$ (Fig. 317).

If the cross-section is variable we suppose the solid divided up into a number of thin slices by parallel planes at equal intervals $\delta x$. Let A be the area of a section at a distance $x$ from one end as in Fig. 318. The graph of A against $x$ is shown in Fig. 319. Throughout a slice the cross-section has nearly the same value and hence

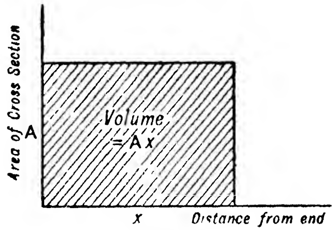

FIG. 317.

the volume of the cross-section between $x = OM$ and $x = ON$.

$$\simeq MP \times ON = \text{area of PMNK.}$$

Hence the total volume of the solid between the sections at $x = a$ and $x = b$ is approximately the sum of the areas under the dotted line in Fig. 319. By taking the thickness of the

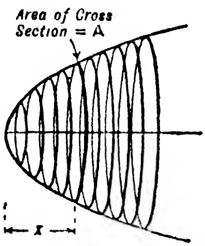

FIG. 318.

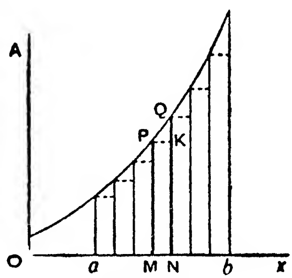

FIG. 319.

slices smaller and smaller we see that the total volume from $x = a$ to $x = b$ must be exactly equal to the area under the graph of A from $x = a$ to $x = b$. In symbols,

$$\text{Volume from } x = a \text{ to } x = b \text{ is equal to } \int_{x=a}^{x=b} A\,dx.$$

As in the case of velocity and time the volume obtained is

only numerically equal to the area, or is represented by the area, i.e. its units are not those of an area.

*Example.*—To find the volume of a right circular cone of height $h$ and base radius $r$.

Fig. 320 shows the cone with its axis OB horizontal. A section perpendicular to the axis by a plane at a distance $x$ from O is also shown. Let this section have radius $y$.

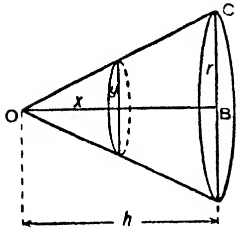


FIG. 320.

From Fig. 320 we see by similar triangles that

$$\frac{y}{x}=\frac{r}{h}$$

$$\therefore\ y=\frac{rx}{h}.$$

Hence the area of the section of radius $y$ is $\pi y^2=\dfrac{\pi r^2x^2}{h^2}$.

Therefore the volume of the cone equals the area under the graph of $\dfrac{\pi r^2x^2}{h^2}$ against $x$ from $x=0$ to $x=h$, which is the integral of $\dfrac{\pi r^2x^2}{h^2}$ between the limits 0 and $h$.

$$\therefore\ \text{Volume of cone}=\int_0^h\frac{\pi r^2x^2}{h^2}\,dx$$

$$=\left[\frac{\pi r^2}{h^2}\cdot\frac{x^3}{3}\right]_0^h$$

$$=\frac{\pi r^2}{h^2}\cdot\frac{h^3}{3}-0$$

$$=\tfrac{1}{3}\pi r^2h.$$

**Work done by a variable force**

If a constant force F lb. wt. moves a body $x$ ft. it does work F$x$ ft. lb. wt. In this case the graph of the force F against

the distance $x$ is a straight line, as in Fig. 321, and the work done $Fx$ is the area shaded.

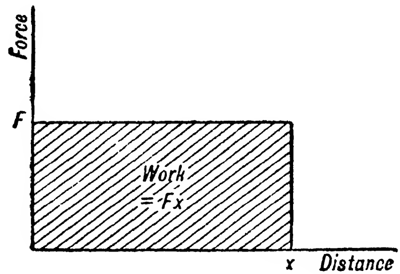


Fig. 321.

Now, if F is variable, the graph of F against $x$ is a curve, as shown in Fig. 322. But in any short distance MN the force F is very nearly constant so that the work done is nearly equal to PM $\times$ MN or the area of the rectangle PMNK. Hence, by the same arguments as in finding the volume on p. 389,

the work done by F as the body moves from $x=a$ to $x=b$,

$=$ the area under the graph of F from $x=a$ to $x=b$

$$=\int_a^b \mathrm{F}\,dx.$$

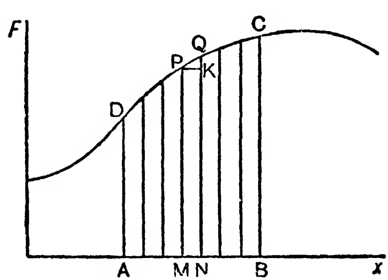


Fig. 322.

**Work done by an expanding gas**

Let the gas be contained in a cylinder of cross-section S ft.$^2$ Let its pressure be $p\,\dfrac{\text{lb. wt.}}{\text{ft.}^2}$ and its volume $v$ ft.$^3$ Suppose that as the gas expands it pushes a piston along the cylinder. Then the force on the piston is $p$S lb. wt. Now, if the volume of the gas increases by a small amount $\delta v$ ft.$^3$, the piston moves $\dfrac{\delta v}{S}$ ft. In this movement the force on the piston is nearly constant and hence it does work which is nearly

$$pS \text{ lb. wt.} \times \frac{\delta v}{S} \text{ ft.} = p\delta v \text{ ft. lb. wt.}$$

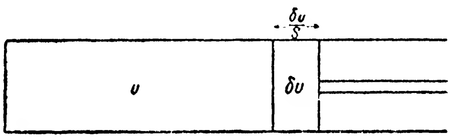


FIG. 323.

But $p\delta v$ is nearly the area of a strip under the $p$, $v$ graph from $v$ to $v+\delta v$, and hence, by the same arguments as before, the work done by a gas as it expands from a volume $v_1$

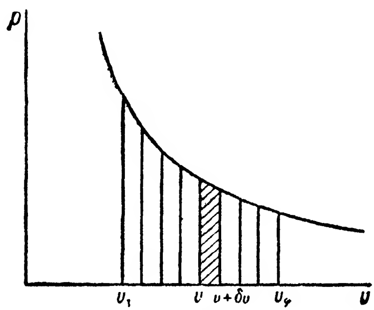


FIG. 324.

to a volume $v_2$ equals the area under the $p$, $v$ graph from $v=v_1$ to $v=v_2$, or $\int_{v_1}^{v_2} pdv$.

## Approximate evaluation of integrals

We have seen in Part I that the area under a curve can be found approximately by either of the following rules:

(*a*) The mid-ordinate rule.

Divide the area into a number of strips by ordinates at equal intervals and measure the ordinates at the middle of each interval. Then, if the strips are of width $h$:

Area under the curve $\simeq h \times$ sum of mid-ordinates.

(*b*) Simpson's rule.

Divide the area into $2n$ strips by ordinates having lengths $y_1, y_2, \ldots y_{2n+1}$ at equal intervals $h$. Then area under the curve

$$\simeq \tfrac{1}{3}h[y_1 + y_{2n+1} + 4(y_2 + y_4 + \ldots + y_{2n}) + 2(y_3 + y_5 + \ldots + y_{2n-1})]$$

$= \tfrac{1}{3}h$[sum of end ordinates + 4(sum of even ordinates) + 2(sum of odd ordinates)].

We have already used the mid-ordinate rule on p. 374 in finding the area under a velocity-time graph. Either of these rules can be used to approximate to the value of any quantity such as a volume or work which can be represented by the area under a curve. In other words these rules enable us to integrate approximately.

*Example.*—The force F lb. wt. on a body when it has moved $x$ ft. is given by the following table. Find the work done by the force when the body moves 12 ft.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| F | .. | 14 | 15·5 | 17·3 | 20·0 | 23·5 | 27·5 | 38·5 |
| $x$ | .. | 0 | 2 | 4 | 6 | 8 | 10 | 12 |

Using Simpson's rule :

$$\begin{aligned}\text{Work done} &= \text{area under F, } x \text{ graph from } x=0 \text{ to } 10\\ &\simeq \tfrac{2}{3}\{14+38\cdot5+4(15\cdot5+20\cdot0+27\cdot5)\\ &\qquad +2(17\cdot3+23\cdot5)\}\\ &= \tfrac{2}{3}\{52\cdot5+4(63\cdot0)+2(40\cdot8)\}\\ &\simeq 257 \text{ ft. lb. wt.}\end{aligned}$$

**Average or mean ordinate of a graph**

Suppose the area under a graph of $y$ against $x$ from $x=a$ to $b$ is divided into $n$ strips by ordinates at equal intervals, and that the lengths of the mid-ordinates of the intervals are $z_1, z_2, z_3 \ldots z_n$.

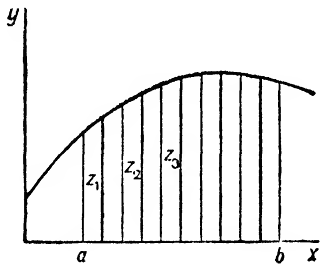


FIG. 325.

Then by the mid-ordinate rule :

$$\begin{aligned}\text{Area under curve} &\simeq \text{width of strip} \times \text{sum of mid-ordinates}\\ &\simeq \frac{b-a}{n} \times (z_1+z_2+z_3+\ \ldots\ +z_n)\\ &\simeq (b-a) \times \left(\frac{z_1+z_2+z_3+\ \ldots\ +z_n}{n}\right)\\ &\simeq (b-a) \times \text{average of mid-ordinates.}\end{aligned}$$

$$\therefore \text{Average of mid-ordinates} \simeq \frac{\text{area under curve}}{\text{base of the area}}.$$

This becomes more and more accurate the larger the number of ordinates we take. For this reason we call the ratio of the area to the base the average or mean ordinate of the graph from $x=a$ to $b$.

For example, the average force acting on the body considered on p. 393,

$$=\frac{257 \text{ ft. lb. wt.}}{12 \text{ ft.}} \simeq 21\cdot 4 \text{ lb. wt.}$$

Also the average velocity of the electric train considered on p. 375

$$=\frac{12\cdot 64 \text{ miles}}{16 \text{ min.}}=\frac{12\cdot 64\times 60}{16} \text{ m.p.h.}=47\cdot 4 \text{ m.p.h.}$$

Since the area under the graph of $y$ against $x$ from $x=a$ to $b$ is $\int_a^b y\,dx$ we have the following result:

Mean value of $y$ from $x=a$ to $x=b$ is equal to $\dfrac{\int_a^b y\,dx}{b-a}$.

## Exercise LI

1. A field ABCD has straight sides along AB, BC and AD, and the angles at A and B are right angles. The off-sets of the curved side CD from AB are given by :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Distance from A | .. | 0 | 30 | 60 | 90 | 120 | 150 | 180 yds. |
| Off-set | .. . | 70 | 110 | 130 | 170 | 160 | 130 | 120 yds. |

Find the area of the field in acres to the nearest $\frac{1}{10}$th acre.

2. The area A sq. ft. of the cross-section of a reservoir by a horizontal plane at $y$ ft. from the bottom is given by :

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $y$ | .. | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
| A | .. | 10,500 | 12,820 | 15,000 | 17,040 | 18,940 | 20,000 | 21,400 |

Find the volume of water of reservoir in gallons when the water is (*a*) 20 ft. deep, (*b*) 30 ft. deep.

3. The following table gives the area A sq. ft. of the cross-section of a log at $x$ ft. from one end. Find the volume of the log.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x$ | .. | 0 | 3 | 6 | 9 | 12 | 15 | 18 |
| A | .. | 1 | 1·1 | 1·2 | 1·32 | 1·6 | 2·3 | 3·6 |

4. During the expansion stroke of an engine the thrust of the piston is given by :

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Clearance of piston $x$ in.— | | | | | | | | | | | | |
| 0·77 | 0·83 | 1·02 | 1·30 | 1·66 | 2·07 | 2·46 | 2·83 | 3·16 | 3·43 | 3·62 | 3·74 | 3 77 |
| Thrust T lb. wt.— | | | | | | | | | | | | |
| 500 | 455 | 340 | 242 | 172 | 126 | 99 | 82 | 70 | 62 | 58 | 55 | 55 |

Find the work done by the force T during the stroke, and find the average thrust on the piston during the stroke.

5. The pressure and volume of a gas are given by :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $p$ lb./ft.$^2$ | .. | 100 | 44 | 28·6 | 21 | 16·6 | 13·8 |
| $v$ ft.$^3$ .. | .. | 50 | 100 | 150 | 200 | 250 | 300 |

Find the work done by the gas in expanding from 50 to 300 cu. ft. Find also the mean pressure during the expansion.

6. The area A sq. ft. of the vertical cross-section of a railway cutting at $x$ ft. from a certain point on the railway is given by :

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| A | .. | 180 | 190 | 210 | 235 | 265 | 297 | 332 | 375 | 425 | 460 | 475 |
| $x$ | .. | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 |

Find the volume of earth to be excavated from $x=0$ to $x=200$, and deduce the mean area of a vertical section.

7. The acceleration of a body moving in a straight line is $\frac{dv}{dt}$ where $v$ is the velocity after a time $t$. What does the area under the acceleration-time graph from $t=a$ to $t=b$ represent ?

The acceleration of a body is given by the following table :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Acceleration $\frac{dv}{dt}$ (ft./sec.$^2$) | 0·3 | 0·38 | 0·52 | 0·60 | 0·90 | 1·12 | 1·40 |
| Time (sec.) .. .. | 0 | 4 | 8 | 12 | 16 | 20 | 24 |

If the body is moving at 10 ft./sec. at $t=0$, find its velocity at $t=24$.

8. Use the table in Question 2 to construct a graph of the volume of water in the reservoir against the depth of the water.

**9.** Use the table in Question 7 to construct a graph of the velocity of the body against the time.

**10.** Show that the area of the cross-section of the cone, formed by revolving the line $y=kx$ from $x=0$ to $h$ about O$x$, by a plane perpendicular to O$x$ at a distance $x$ from the origin is $\pi k^2x^2$. Deduce that the volume of the cone is $\int_0^h \pi k^2x^2dx$. Evaluate this and hence find a formula for the volume of a cone of height $h$ and base radius $r$.

**11.** The force required to extend a spring $x$ in. is $sx$ lb. wt. where $s$ is a constant, the stiffness of the spring. Show that the work done in increasing the extension from $a$ in. to $b$ in. is $\frac{1}{2}s(b^2-a^2)$ in. lb. wt.

**12.** When a certain gas is expanding adiabatically the pressure $p$ lb. wt./ft.$^2$ and volume $v$ ft.$^3$ are related by $pv^{1\cdot 3}=200$. Find $p$ in terms of $v$ and hence find the work done by the gas in expanding from $v=\frac{1}{2}$ to $v=1$, and the mean value of $p$ during the expansion.

**13.** A chain of length 10 ft., which weighs $\frac{1}{2}$ lb. wt. per foot, is coiled on the floor. One end is lifted vertically. What force is required to do this when $x$ ft. of the chain are off the floor? Show that the work done in lifting the whole chain just clear of the floor is $\int_0^{10} \frac{1}{2}xdx$ ft. lb. wt. Hence find the work done.

**14.** If the relation between the pressure and volume of a gas is $pv^n=c$ where $c$ is a constant and $n$ is not equal to 1, find the work done as the gas expands from a volume $v_1$ to a volume $v_2$, and show that it equals $\frac{p_1v_1-p_2v_2}{n-1}$, where $p_1$, $p_2$ are the pressures when the volume is $v_1$ and $v_2$ respectively.

**15.** Find the mean value of the ordinate of $y=x^2$ from $x=0$ to $a$.

**16.** If O, A, B are the points (0, 0) (2, 4), (3, 0) respectively, find the mean ordinate of the graph OAB.

**17.** Show by graphs that the areas under $y=\sin^2 x$ and $y=\cos^2 x$ from $x=0$ to $\pi$ are equal. Using the identity $\sin^2 x+\cos^2 x=1$, find the value of each area and deduce that the mean value of $\sin^2 x$ from $x=0$ to $x=\pi$ is $\frac{1}{2}$.

**18.** Find by integration the volume of a cone having a height of 12 in. and a base radius of 4 in. Do not quote the formula.

**19.** If a pyramid has a height $h$ and a base of area A, show that the area of a cross-section parallel to the base at distance $x$ from the vertex is $\dfrac{Ax^2}{h^2}$. Hence prove that its volume is $\frac{1}{3}Ah$.

**20.** Show that the volume of a frustum of a cone having end radii 2 in. and 3 in. respectively and length 8 in. is $\displaystyle\int_{16}^{24} \frac{\pi x^2}{64} dx$, and hence evaluate it.

# ANSWERS

## EXERCISE I (Page 4)

1. (*a*) $3\sqrt{3}=5{\cdot}20$ ; (*b*) $3\sqrt[3]{2}=3{\cdot}78$ ; (*c*) $4\sqrt{5}=8{\cdot}95$ ; (*d*) $7\sqrt{3}=12{\cdot}1$ ; (*e*) $12\sqrt{2}=17{\cdot}0$.
2. (*a*) $\dfrac{\sqrt{2}}{2}=0{\cdot}707$ ; (*b*) $\dfrac{\sqrt{5}}{10}=0{\cdot}224$ ; (*c*) $2\sqrt{3}=3{\cdot}46$ ; (*d*) $2\sqrt{5}=4{\cdot}47$.
3. (*a*) $7^{\frac{1}{2}}$ ; (*b*) $3^{\frac{1}{3}}$ ; (*c*) $\dfrac{1}{2^{\frac{1}{4}}}$ ; (*d*) $(\frac{2}{3})^{\frac{1}{2}}$ ; (*e*) $5^{\frac{7}{2}}$.
4. (*a*) $t^{\frac{1}{2}}$ ; (*b*) $x^{\frac{3}{2}}$ ; (*c*) $y^{\frac{5}{2}}$ ; (*d*) $2m^{\frac{1}{2}}$ ; (*e*) $x^{\frac{2}{3}}y^{\frac{2}{3}}$.
5. (*a*) $32a$ ; (*b*) $27$ ; (*c*) $y^4/x^3$.
6. (*a*) $1/8n^4$ ; (*b*) $3r^2/20$ ; (*c*) $w/W$.
7. The latter.
8. The former.
9. (*a*) $x^{\frac{7}{2}}$ ; (*b*) $x^{-\frac{2}{3}}$ ; (*c*) $x^{-\frac{1}{2}}$ ; (*d*) $x^{1{\cdot}8}$.
10. (*a*) $2$ ; (*b*) $y^{\frac{1}{2}}$ ; (*c*) $3a^{\frac{2}{3}}b^{\frac{1}{3}}$ ; (*d*) $a^{\frac{9}{2}}$.
11. (*a*) $16/81$ ; (*b*) $2r^2$ ; (*c*) $432x^{\frac{25}{12}}$ ; (*d*) $p^2/q$.
12. (*a*) $p^{\frac{2}{3}}$ ; (*b*) $50$ ; (*c*) $a^{\frac{7}{4}}b^{-\frac{15}{8}}$.
13. (*a*) $1/v^2$ ; (*b*) $m^{\frac{1}{2}}/n^{\frac{3}{2}}$ ; (*c*) $\frac{1}{3}q^{\frac{1}{2}}r^{\frac{3n}{2}}$.
14. (*a*) $1/\sqrt{y}$ ; (*b*) $q^3/p^3$ ; (*c*) $\sqrt[3]{y/x}$ ; (*d*) $m/lt^2$.
15. (*a*) $l$ ; (*b*) $\sqrt[4]{x}/\sqrt{y}$ ; (*c*) $\sqrt{3/pq}$.
16. $y=2x^{\frac{1}{4}}$.
18. $r=\sqrt{V/\pi h}$.

## EXERCISE II (Page 9)

1. (*a*) $2x^2-8x$ ; (*b*) $2M^2+3MN$ ; (*c*) $2\omega t+\dfrac{4\pi}{3}$ ; (*d*) $a^3-a^2h$.
2. (*a*) $3x^2+5x$ ; (*b*) $3\theta+7\alpha$ ; (*c*) $6y^2+2ay+2a^2$ ; (*d*) $2r_1l_1$.
3. (*a*) $x^2+a^2+b^2$ ; (*b*) $4p^2$ ; (*c*) $-m^2-2mn-5n^2$.
4. (*a*) $2m+2n$ ; (*b*) $-3x+5y$ ; (*c*) $4ab$ ; (*d*) $nt-90$ ; (*e*) $a^2-b^2-ca$.
5. (*a*) $7m+3n$ ; (*b*) $2p-6q$ ; (*c*) $x^2+4ax+3a^2$ ; (*d*) $6{\cdot}8t^2-10{\cdot}6t$.
6. (*a*) $2a^3-5a^2h-ah^2$ ; (*b*) $-4+7\cos\theta$ ; (*c*) $2-7\cos A$ ; (*d*) $\sin x-\cos x$.
7. (*a*) $t^6+2t^7-6t^8$ ; (*b*) $4a+2b-7c$ ; (*c*) $-6x$ ; (*d*) $3\cos x$.
8. (*a*) $6x^2-x-12$ ; (*b*) $l^3-2l^2m+lm^2-2m^3$ ; (*c*) $8-14x+3x^2$ ; (*d*) $2p^3-p^2q-pq^2$.
9. (*a*) $4x^3-14x^2+20x-16$ ; (*b*) $x^4+x^3-6x^2-17x-21$ ; (*c*) $l^2m+lp+lm^2p+mp^2$ ; (*d*) $1+e^2+e^4$.
10. (*a*) $x^2+5x+4$ ; (*b*) $a^2-ab-6b^2$ ; (*c*) $p^2-3pq-4q^2$ ; (*d*) $2y^2+3y-35$ ; (*e*) $6x^3+7x^2-x-2$ ; (*f*) $l^3+l^2m-2lm^2$.
11. (*a*) $7+2x+x^2$ ; (*b*) $4y^2+12y+9$ ; (*c*) $p^2-4pq+4q^2$ ; (*d*) $4a^2r^2-20ars+25s^2$.
12. (*a*) $m^2-n^2$ ; (*b*) $\dfrac{r^2}{s^2}-1$ ; (*c*) $a^2-b^2-2bc-c^2$.

13. (*a*) 28561; (*b*) 17956; (*c*) 9604; (*d*) 2209.
14. (*a*) $\frac{7}{6a}$; (*b*) $\frac{3x}{14}$; (*c*) $\frac{a^2+b^2}{a^2b}$; (*d*) $\frac{1-2y}{y}$.
15. (*a*) $\sqrt{2}-1=0{\cdot}414$; (*b*) $\frac{5}{2}(4+\sqrt{6})=16{\cdot}12$; (*c*) $\frac{1}{3}(5+\sqrt{7})=2{\cdot}55$.
16. 0·02502. 17. 0·02944.
18. (*a*) $x^2+2xy+y^2$ or $(x+y)^2$; (*b*) $4ab$.
19. (*a*) $-2xy-y^2$ or $-y(2x+y)$; (*b*) $1-\sin^2\theta$ or $\cos^2\theta$.
20. (*a*) $8+2\sin^2 x$; (*b*) $\frac{1}{2}wx^2-\frac{1}{2}wxl$ or $\frac{1}{2}wx(x-l)$.
21. (*a*) $3+3x-x^2$; (*b*) $x^3-5$. 22. $8y^3+36y^2+54y+27$.
23. $a^3-3a^2b+3ab^2-b^3$, $k^3-3k^2+3k-1$.

## EXERCISE III (Page 16)

1. (*a*) 3; (*b*) $4\frac{1}{2}$; (*c*) $1\frac{2}{3}$; (*d*) $1\frac{1}{2}$; (*e*) $\frac{3}{4}$; (*f*) 2.
2. (*a*) $p=3(q+r)$; (*b*) $q=\frac{1}{2}p$; (*c*) $R=5r$; (*d*) $h=\frac{S}{2\pi r}-\frac{r}{2}$; (*e*) $E=\frac{4Wl^3}{bd^3y}$; (*f*) $v=\frac{uf}{u-f}$.
3. (*a*) 1/9; (*b*) 2; (*c*) 8/9; (*d*) $-0{\cdot}171$.
4. (*a*) $p=\frac{2}{3}D^2$; (*b*) $p=\frac{100A}{100+rt}$; (*c*) $g=\frac{4\pi^2 l}{T^2}$; (*d*) $h=\frac{r}{3}+\frac{V}{\pi r^2}$; (*e*) $C=\frac{1}{4\pi^2 lf^2}$; (*f*) $r=\frac{h^2+y^2}{2h}$.
5. $2s-2a$. 6. (*a*) $k=\frac{m-x}{1+mx}$; (*b*) $m=\frac{k+x}{1-kx}$; (*c*) $x=\frac{m-k}{1+km}$.
7. $t=\frac{n_1d_1-n_2d_2}{n_2-n_1}$. 8. $\theta_m=\frac{x\theta_1}{\theta_1-y}$. 10. (*c*).
11. $\frac{1}{2\pi\sqrt{LC}}$. 12. $\frac{35344\ EI}{N^2L^4}$. 13. $(Tc^2-T^2)/2Tc$.
14. $\frac{R_1{}^2R_4{}^2-R_2{}^2R_3{}^2}{R_2{}^2L_3{}^2-R_1{}^2L_4{}^2}$. 15. $6\frac{1}{6}$, $5\frac{5}{6}$ in. 16. 22·6 in.
17. 5 min. 18. 21, 44. 19. 600.
20. 12 ohm. 21. 157·5 lb. 23. $-40$.
24. 144; $\frac{12n}{7}$. 25. 32·5 knots; $\frac{20n+3000}{n-100}$ knots.
26. 2·3 mm. 27. 23·5 miles per gallon.
28. £7 10*s*. 0*d*.; £5 10*s*. 0*d*.; 160 units.

## EXERCISE IV (Page 22)

1. (*a*) ft./sec., velocity; (*b*) ft., distance; (*c*) ft./sec., velocity; (*d*) ft., distance.
2. (*a*) amps; (*b*) watts; (*c*) watts.
3. (*b*), (*b*). 4. ft. lb. wt.
5. 1 lb. wt. $=32{\cdot}19$ lb. mass $\times\frac{\text{ft.}}{\text{sec.}^2}$.
6. $\frac{\text{1 lb. mass. ft.}}{\text{sec.}^2}$, 1 dyne $=\frac{\text{1 gram. cm.}}{\text{sec.}^2}$, 1 kilogram wt. $=981\times10^3$ dynes.
7. $4{\cdot}45\times10^5$ dynes. 8. lb. wt./in.$^2$.

## EXERCISE V (Page 25)

1. $6i$ volts.
2. $\frac{1}{2}x^2$.
3. $16t^2$ ft.
4. $\pi(2r+1)$ in.$^2$.
5. $(n+430)$ pence.
6. $A^{\frac{3}{2}}$.
7. $\frac{1}{2(y+3)}$.
9. $\frac{1}{4}u^2-u$.
10. $A=\pi r^2$; $r=\sqrt{\frac{A}{\pi}}$.
11. $t=\frac{pv}{R}-273$.
12. $\sqrt{a^2+b^2}$.
13. (a) $\frac{1}{4}\pi h^3$; (b) $2\pi r^2$.
14. $\frac{ab}{\sqrt{a^2+b^2}}$.
15. $\frac{\sqrt{R^2-Z^2}}{ZRC}$.
16. (a) $\frac{4y+5}{y+3}$; (b) $\frac{5-3x}{4-x}$.
17. $\frac{v(u-r)}{u(v-r)}$.

## EXERCISE VI (Page 29)

1. $x=\frac{1}{3}$, $y=\frac{5}{3}$.
2. $x=14\frac{3}{19}\simeq 14\cdot16$, $y=-4\frac{17}{19}\simeq -4\cdot89$.
3. $x=\frac{1}{50}=0\cdot02$, $y=1\frac{3}{50}=1\cdot06$.
4. $x=\frac{1}{3}$, $y=\frac{1}{7}$.
5. $x=15\cdot9$, $y=18\cdot5$.
6. $x=1\cdot58$, $y=1\cdot45$.
7. $x=0\cdot467$, $y=-0\cdot795$.
8. $x=1\cdot09$, $y=1\cdot30$.
9. $p=9$, $q=4$.
10. $a=0\cdot0442$, $b=1\cdot047$.
11. $m=3$, $c=-8$
12. $x=\frac{5}{4}$, $y=\frac{5}{8}$.
13. $x=\frac{rE}{R(r+s)+rs}$, $y=\frac{sE}{R(r+s)+rs}$.
14. $P=23\cdot2$, $N=28\cdot9$.
15. $T_1=50+25\sqrt{3}\simeq 93\cdot3$, $T_2=50\sqrt{3}-25\simeq 61\cdot6$.
16. $k=\frac{e(D-d)}{DdR}$.
19. $f=\frac{1}{2\pi\sqrt{LC}}$.
21. $a=(BC-1)/A$, $k=A(C-1)/(BC-1)$, $k'=A(B-1)/(BC-1)$.
23. $V^2=(a_1^2-a_2^2)/(T_1^2-T_2^2)$, $d^2=(a_1^2T_2^2-a_2^2T_1^2)/(T_1^2-T_2^2)$.
24. $h=\frac{\sqrt{3}}{2}d$.

## EXERCISE VII (Page 38)

1. Min. $y=-13\cdot75$; $2\cdot16$, $-0\cdot66$.
2. Max. $R=5$, Min. $R=-3$; $-1\cdot64$, $0\cdot17$, $1\cdot81$.
3. $1\cdot36$, $-0\cdot473$.
4. Min. $t=6\cdot92$ at $m=1\cdot52$.
5. Max. $z=97\cdot4$ at $y=-3\cdot65$, min. $z=-97\cdot4$ at $y=3\cdot65$.
6. $1\cdot56$, $0\cdot403$.
7. Max. $p=5\cdot33$.
8. $-1\cdot7$, $0\cdot3$, $2\cdot2$, $2\cdot16$.
11. Max. $y=2\cdot24$; $74°$.
20. $-3$, $5$.
21. $-\frac{1}{2}$, $4$.
22. $\frac{4}{3}$, $\frac{7}{3}$.
23. $2$, $-\frac{12}{5}$.
24. $\frac{5}{9}$, $-\frac{160}{9}$.
25. $-0\cdot8$, $200$.
26. $3y-4x=1$.
27. $5y+8x=68$.
28. $x+y=3$.
31. $23\cdot1$.
32. $x=-3$, $y=8$; $x=\frac{5}{3}$, $y=\frac{16}{9}$.
33. $40\cdot6$, $69\cdot0$.
34. $1430$.
35. $4\cdot025$.

## EXERCISE VIII (Page 42)

1. (a) $a(a+b)$; (b) $x(x-y)$; (c) $(a+h)(a-h)$.
2. (a) $q(2p-3q)$; (b) $(p+q)(p-q)$; (c) $(2l+m)(2l-m)$.
3. (a) $(x+3)(x-3)$; (b) $(11+r)(11-r)$; (c) $240\times 2$.
4. (a) $(3c+4r)(3r-4r)$; (b) $8(k+5)(k-5)$; (c) $(\frac{1}{2}a+r)(\frac{1}{2}a-r)$.
5. (a) $(pq+ab)(pq-ab)$; (b) $p^2(q+b)(q-b)$;
   (c) $5(2pq+3ab)(2pq-3ab)$.
6. (a) $3(x-1)(x+1)$; (b) $2x(x-l)$.
7. (a) $(l+2)(l-2)(l^2+4)$; (b) $l^2(l+4)(l-4)$; (c) $3r(2a+r)$.
8. (a) $(2x+3y)(2x-3y)(4x^2+9y^2)$; (b) $4(a+b)c$.
9. (a) $\left(\frac{t}{T}+1\right)\left(\frac{t}{T}-1\right)$; (b) $(r^n+1)(r^n-1)$; (c) $\left(k+\frac{3}{m}\right)\left(k-\frac{3}{m}\right)$.
10. (a) 255; (b) 256; (c) 1·1808.
11. (a) 1200; (b) 1725; (c) 60736.
12. 7·11. 13. $5{\cdot}04\times 10^{12}$.
15. (a) $(a-2)(x-1)$; (b) $(l+m)(l+n)$.
16. (a) $(k-l)(a-b)$; (b) $(t-3s)(t+1)$.
17. (a) $(a+b+c)(a+b-c)$; (b) $(a-b+c)(a-b-c)$.
18. (a) $(a+b-c)(a-b+c)$; (b) $(2k+l+3m)(2k+l-3m)$.
19. (a) $(x+1)(x+4)$; (b) $(x-1)(x-4)$; (c) $(x+1)(x-4)$.
20. (a) $(x+5)(x+18)$; (b) $(x-5)(x-18)$; (c) $(x-5)(x+18)$.
21. (a) $(p+7)(p+9)$; (b) $(p+7)(p-9)$; (c) $(p-7)(p+9)$.
22. (a) $(y+3)(y+5)$; (b) $(y-3)(y+5)$; (c) $(y+3)(y-5)$.
23. (a) $(x+4)(x-1)$; (b) $(x+7)(x-4)$; (c) $(x-7)(x+4)$.
24. (a) $(l-2)(l+3)$; (b) $(l+2)(l-3)$; (c) $(l-1)(l-6)$.
25. (a) $(m-7)(m+2)$; (b) $(m+7)(m-2)$; (c) $(m-1)(m\times 14)$.
26. (a) [illegible]; (b) $(p-q)(p-2q)$; (c) $(p-2q)(p+q)$.
27. (a) [illegible]; (b) $(l-2m)(l+m)$; (c) $(l-6m)(l+5m)$.
28. (a) $(pq-2)(pq-3)$; (b) $(p-2q)(p-3q)$; (c) $(pq-3)(pq+2)$.
29. $xy\,(y-x)$. 30. $-2x\,(a-x)$.
31. $(l-x)(l+x)(x+n)$. 32. $5t\,(2r-t+2)$.
33. (a) $(1-\cos\theta)(1+\cos\theta)$; (b) $(\cos A-\sin A)(\cos A+\sin A)$;
   (c) $(4\cos A+3\sin A)(2\cos A+\sin A)$.
34. (a) $(2\tan\theta-1)(2\tan\theta+1)$; (b) $(1+\sin\theta)(1+2\cos\theta)$;
   (c) $(\cos x-\sin x)(\cos x+\sin x)$, after putting $\sin^2 x+\cos^2 x=1$.

## EXERCISE IX (Page 47)

1. $(2x+1)(x+1)$. 2. $(3t+4)(t+1)$. 3. $(3t+2)(t+2)$
4. $(4y+1)(y-2)$. 5. $(2m+n)(m-2n)$. 6. $(4x+1)(x-4)$.
7. $(3z-2)(2z-1)$. 8. $(3l+1)(l-2)$. 9. $(3+5k)(1+k)$.
10. $x-7$. 11. $y+1$. 12. $a+b$.
13. $4p+q$. 14. $x-1$; $-2$. 15. $3x^2+x+4$, 5.
16. $4y+1$, $10y$. 17. $\frac{1}{3}t^2+\frac{1}{9}t-\frac{17}{27}$, $4\frac{10}{27}$. 20. $(y-1)(y^2+y+1)$.
21. $(x+a)(x^2+ax+a^2)$. 22. $(3+p)(9-3p+p^2)$.
23. $20(l+2)(l^2-2l+4)$. 24. $x^2-lx-l^2$; $x(x-l)(x^2-lx-l^2)$.
25. 9·05 cu. in.; 0·0002 in.

## EXERCISE X (Page 52)

1. $b^2/x$. 2. $4n^2/3lm$. 3. $\frac{2}{5}r^2$.
4. $24x/r$. 5. $(x-y)/(x+z)$. 6. $\frac{1}{4}(p+2)$.
7. $a/c$. 8. $(x+3)/(x+1)$. 9. $(r-s)/(r+s)$.
10. $1/(p+2q)$. 11. $(z+1)/(z+4)$. 12. $x/(x+5)$.
13. $y^2-y-6$. 14. $c^2/(a+2c)$. 15. $(x+3)/(x+1)$.
16. $r^2-rx$. 17. $pq/(p+q)$. 18. $-6/ab$.
19. $(rs-1)/(rs-s^2)$. 20. $(m+n)/(m-n+2)$.
21. $1/(2x-1)$. 22. $y/(y-1)$.
23. $(m-n)/(2m+3n)$. 24. $(2l-1)/(l-2)$.
25. $\dfrac{1}{(x+3)(x+4)}$. 26. $\dfrac{3+t}{t(3-t)}$. 27. $\dfrac{4c-5a}{6abc}$.
28. $\dfrac{m+4}{m^2+m}$. 29. $\dfrac{1}{l-l^2}$. 30. $\dfrac{16y}{1-9y^2}$.
31. $\dfrac{3a^2}{a^2-b^2}$. 32. $\dfrac{1}{y+2}$ 33. $\dfrac{4rs}{(r^2-s^2)^2}$.
34. $-\dfrac{2}{x}$. 35. $\dfrac{1}{2z^2-z-1}$. 36. $\dfrac{6pq}{p^2-4q^2}$.
37. $\dfrac{4}{(x-1)(x-2)(x+3)}$. 38. $-\dfrac{2}{x(x-1)(2x-1)}$.
39. $\dfrac{2}{(t-s)(3t+1)}$. 40. $-\dfrac{xy}{(x-y)(x^2+y^2)}$.
41. (*a*) $x$ ; (*b*) does not simplify.
42. (*a*) does not simplify ; (*b*) 2.
43. (*a*) does not simplify ; (*b*) $m+n$.
44. (*a*) $2/mn$ ; (*b*) does not simplify.
45. (*a*) $(k-l)/(k-l)$ ; (*b*) does not simplify.
50. 1. 51. $-\frac{5}{3}$. 52. 1. 53. 2.
54. $\frac{1}{6}Wab(a+2b)$. 56. *No*. 57. $\frac{1}{2}(a^2+b^2)$.
58. $\dfrac{rr_1+rr_2}{r+r_1+r_2}$, $\dfrac{rr_1+rr_2+r_1r_2}{r_1+r_2}$.

## EXERCISE XI (Page 67)

1. $\pm\sqrt{\frac{3}{2}} \simeq \pm 1{\cdot}225$. 2. $\pm\frac{1}{6}$. 3. $\pm\sqrt{20} \simeq \pm 4{\cdot}472$.
4. $\frac{1}{3}$ or $\frac{5}{3}$. 5. 1 or 2. 6. $-1$ or 5.
7. $-1$ or $\frac{3}{2}$. 8. $-\frac{8}{3}$ or $\frac{1}{4}$. 9. $-\frac{3}{2}$ or $-\frac{1}{2}$.
10. $-\frac{5}{7}$ or 1. 11. $-2$ or 3. 12. $-11$ or 2.
13. $-2$ or $-\frac{11}{2}$. 14. $-4$ or $-1$. 15. $-1$ or 5.
16. $-4$ or $-2$. 17. $-3$ or 8. 18. $-8$ or $-1$.
19. 9. 20. $\frac{1}{9}$.
21. to 30. Answers same as 5–14. 31. $-2{\cdot}56$ or $1{\cdot}56$.
32. $-2{\cdot}281$ or $-0{\cdot}219$. 33. $-2{\cdot}52$ or $1{\cdot}19$.
34. $-0{\cdot}640$ or $0{\cdot}39$. 35. $0{\cdot}13$ or $3{\cdot}87$.
36. $-0{\cdot}395$ or $4{\cdot}27$. 37. $-0{\cdot}24$ or $2{\cdot}95$.
38. $-0{\cdot}0865$ or $2{\cdot}47$. 40. $-\frac{1}{2}l \pm \sqrt{\frac{1}{4}l^2-k^2}$.

**42.** 0·551 or 5·449. **43.** $-2·680$ or 0·186.
**44.** $-1·14$ or 6·14. **45.** $-22·25$ or 2·25.
**46.** $\dfrac{-a\pm\sqrt{a^2+4bc}}{2c}$. **47.** $\frac{1}{2}(c\pm\sqrt{c^2-4fc})$.
**48.** $r=\dfrac{W}{2\pi slt}-\frac{1}{2}t,\ t=-r\pm\sqrt{\dfrac{W}{\pi sl}+r^2}$.

## EXERCISE XII (Page 70)

**1.** $\frac{1}{2}$. **2.** 1·55.
**3.** (*a*) after 2 or $\frac{17}{4}$ sec.; (*b*) after $\frac{1}{4}$ or 5 sec.
**4.** 9 ft. 6 in., 2 ft. 6 in. **5.** 3·804.
**6.** 50, 2·45; $\frac{1}{2}\{\sqrt{(a+b)^2+0·04ab}-(a+b)\}$.
**7.** 0·44; $\{\sqrt{v^4+4g^2l^2}-v^2\}/2g$.
**8.** 5·98 in. **9.** (*a*) 11·86; (*b*) $\frac{1}{2}$.
**10.** $a=\frac{1}{2}\left(1-\dfrac{1}{\sqrt{3}}\right)l\simeq 0·21\ l$.
**11.** $L=\dfrac{1}{4\pi^2n^2C}\pm\dfrac{\sqrt{Z^2-R^2}}{2\pi n}$; 0·1434, 0·0592, 0·4160.
**12.** 51·06, $v=\left\{t+\sqrt{t^2-2d\left(\dfrac{1}{f}+\dfrac{1}{f'}\right)}\right\}\Big/\left(\dfrac{1}{f}+\dfrac{1}{f'}\right)$.
**13.** $l(\sqrt{n^2+n}-n)$ where $n=w_b/w_t$.

## EXERCISE XIII (Page 77)

**1.** (*a*) $-0·472$, 8·472; (*b*) 4, 4.
**2.** 9. **3.** $-2$, $-1$. **4.** 2; $-4$, $-3$.
**5.** $l\sqrt{1\pm\sqrt{8/15}}\simeq 1·315\ l$ or $0·519\ l$.
**6.** $-3·64$, $-0·64$, 4·28. **7.** 0·782.
**8.** 6·53. **9.** 7 ft. 10 in.

## EXERCISE XIV (Page 85)

**1.** 0·1512. **2.** 6·767. **3.** 1552. **4.** 0·06908.
**5.** 0·3375. **6.** 0·01394. **7.** 14·37. **8.** 1·868.
**9.** 0·9024. **10.** 0·2460. **11.** 5780. **12.** 0·3267.
**13.** 0·004914. **14.** 1·722. **15.** 0·04455. **16.** 2·467.
**17.** 0·9942. **18.** 0·8373. **19.** 128·7. **20.** 0·3675.
**21.** 1·63, $-0·63$.
**22.** 2, 5, 0·30, 0·78, 0·95; $\bar{1}·30$, 2·78, $\bar{2}·95$.
**27.** 0·76, 2·99. **28.** 2·92, 4·30.
**29.** 1·79, $-1·79$, 2·48. **30.** 5·21.

## EXERCISE XV (Page 92)

1. $3{\cdot}775\times10^{-8}$.
2. 7907.
3. 30·96.
4. 400.
5. $2{\cdot}156\times10^{9} gm/cm^{2}$.
6. 2355.
7. 0·99.
8. 75·8, 37.
9. 0·001296.
10. 1745.
11. 17·2.
13. 19350 ; 8·2.
15. 174·5, 95·13 ft./min.
16. 1821.
17. $l=13.71(EI/wN^2)^{\frac{1}{4}}$.
18. $t=(c/v)^8$.
19. $v_1=v_2(T_2/T_1)^{\frac{1}{\gamma-1}}$.
20. $A=0{\cdot}312\ I^{1{\cdot}22}$.
21. $R=\left(1-\frac{E}{100}\right)^{-4}$.
22. $n=\frac{\log C-\log p}{\log v}$.
23. $L=0{\cdot}33\ Q^{0{\cdot}98}\ H^{-1{\cdot}44}$.
24. $r=100\left\{1-\left(\frac{A}{P}\right)^{\frac{1}{n}}\right\}$.
25. $\sqrt{\frac{2\pi^2r^3P}{p}}$ ; $\sqrt[3]{\frac{\rho l^3}{2\pi^2P}}$ ; 0·0097, 0·0154, 0·0284, 0·0450.
26. $p=\frac{b^{16}}{a^{15}}$ ; $v=\left(\frac{a}{b}\right)^{15}$ ; 58·48.

## EXERCISE XVI (Page 98)

1. 2.
2. 1·673.
3. −3·322.
4. −3.
6. 1·3389, 7·519.
7. 0·1786.
8. −1·18.
9. 0·0795.

## EXERCISE XVII (Page 105)

1. $l_1 : l+l_1$.
2. 11 : 2.
3. $\pm\sqrt{5} : 1$.
4. 3 : 1.
5. $\left(1+\sin\frac{\pi}{n}\right)\Big/\left(1-\sin\frac{\pi}{n}\right)$.
6. $b/(a+b)$.
8. $m_1 : m_2=(g+f) ; (g-f)$.
10. (i) 13 ; (ii) $-\frac{5}{3}$ ; (iii) 9.
11. (i) 14/23 ; (ii) 27/23.
12. $R=\frac{71V^2}{90}$, 1818, 5385, 9896, 100·7.
13. $V=2{\cdot}654\sqrt{x}$, 3·75 cu. ft./min. ; 5·75 ; 2·505.
14. $\frac{1}{3}$.

*

15. $1/\sqrt{n}$.

16. $i=\dfrac{25}{R}$.

17. $v_m=r\sqrt{\dfrac{l_m}{l}}$.

18. $z=kxy$ where $k$ is constant ; $k=\frac{1}{2}$ ; 0·4, $2\frac{2}{13}$.

19. $x=kft^2$ where $k$ is constant, $k=\frac{1}{8}$ ; 37·5, 20 ; $t=\sqrt{\dfrac{8x}{f}}$.

20. $V=\frac{25}{24}r^2h$, 18 : 1.

21. $p=\dfrac{kT}{v}$ where $k$ is constant ; $2{\cdot}31\times10^3$ ; 21·644, 4·62.

22. $R=\dfrac{kl}{d^2}$ ; $k=5{\cdot}625\times10^{-6}$ ; (*a*) 2·11 ; (*b*) 6044 cm.

23. $I=khr^4$.

24. $N=klr^4\theta$.

25. $M=kWl^2$.

26. $E=kf^2/s$.

27. $L=kn^2A/l$.

28. $T=k\sqrt{l}/d^2$.

29. $C=5{\cdot}89\times10^{-13}\times\dfrac{(n-1)A}{d}$, 1472 $\mu\mu$F.

30. $F=\dfrac{3{\cdot}82\times10^{-7}Ax}{l}$ where all lengths are in inches, 75 lb. wt.

32. $R=kl^2/w$, 117 ohms.

33. $f\propto\sqrt{d}$.

## EXERCISE XVIII (Page 117)

1. $a=\frac{7}{6}$, $b=-\frac{25}{3}$ ; $78\frac{1}{8}$.

2. $\dfrac{1}{u}=0{\cdot}061-\dfrac{1{\cdot}24}{v}$ ; 16·4 cm.

3. $u=\frac{25}{3}$, $f=\frac{25}{18}$.

4. $3{\cdot}92\times10^{-7}x^{3{\cdot}23}$, 109.

5. $\frac{1}{3}W^{\frac{1}{2}}$, 9 lb.

6. $a=1{\cdot}608$, $b=-0{\cdot}0442$.

7. $R=100+0{\cdot}4t$, $R=100{\cdot}8\{1+0{\cdot}00397(t-2)\}$.

8. $p=24{\cdot}83+0{\cdot}088t$.

9. $W=1{\cdot}07l+1{\cdot}48$.

10. $B=28{\cdot}1d^2+0{\cdot}6$.

11. $e=\dfrac{90}{l}+19{\cdot}1$.

12. $W=\dfrac{415}{x}-3{\cdot}7$.

13. $l=h+\dfrac{225}{h}$.

14. $H=0{\cdot}633\sqrt{d}+\dfrac{0{\cdot}625}{\sqrt{d}}$.

15. $f=\dfrac{3{\cdot}75\times10^8}{7{\cdot}5\times10^3+x^2}$.

16. $pv^{1{\cdot}37}=101$.

17. $t=7{\cdot}95h^{-\frac{1}{2}}$.

18. $I=8{\cdot}9\times10^{-7}V^4$.

## EXERCISE XIX (Page 129)

3. 1·95 in. from A, 0·65 in. from B ; 3·9 in. from A, 1·3 in. from B.

4. $4\frac{1}{3}$.

5. 2·14.

10. 6·4 ft.

## EXERCISE XX (Page 144)

1. (i) Not similar. Sides not proportional; (ii) Similar; (iii) Similar; (iv) Not similar. Angles not equal; (v) Similar; (vi) Similar.
2. 7·5 cm., 5 cm., 6·25 cm., 8·75 cm.
3. BC=9 ft., CD=20 ft. 3 in., DE=6 ft. 9 in., EF=FA=15 ft. 9 in.; area=$430\frac{5}{16}$ sq. ft.
4. 15·3 in., 3·5 in.
5. $\frac{5}{6}$ mile.
6. 76 ft.
7. 4 ft. 8 in. (to the nearest inch).
8. 250,000 miles (approx.).
9. 5 ft. 4 in.
10. 70 ft.
11. 1 : 2.
12. ($2\frac{1}{2}$, 5).
13. R(3, $5\frac{1}{2}$); S($1\frac{1}{2}$, 6).
14. (8, 4·8), (4·8, 8), (1·6, 8), (−1·6, 3·2).
22. 7 : 5; $1\frac{1}{2}$ in. from A and B.
24. 9 in., 24 in.
25. 10 in., 16 in.
26. 6 miles to 1 inch.
27. 91·08 sq. in.
28. 1·304 sq. in.; 20·86 sq. in.
30. $at/(a-b)$.
32. 3 cm.
36. 9 in.

## EXERCISE XXI (Page 163)

1. 54·55 sq. ft.
2. 9·2 in. or 5 ft. 2·8 in.
3. 19 ft. 7 in.
4. 122·5 sq. in.
5. 10 ft. 11·6 in.
6. 1,067 ft.
7. 6·63 cm.
8. 7·26 cm.
9. 0·19 in.
10. $3\frac{1}{8}$ in.
11. 0·27 in., 1·23 in.
12. 5 ft. $2\frac{1}{2}$ in.
15. 1·97 in.
16. 3·8 in.
19. 11·91 cm., 10·67 cm.
20. (i) $\sqrt{c^2-(a-b)^2}$, (ii) $\sqrt{c^2-(a+b)^2}$.

## EXERCISE XXII (Page 178)

1. 10·5 in.
2. (i) $\sqrt{3}a$; (ii) 35° 16′; (iii) 54° 44′.
3. (i) 53° 8′; (ii) 21 ft.; (iii) 36·37 ft.
4. (i) 54° 44′; (ii) 70° 32′.
6. 54° 44′.
7. 10° 20′.
8. 20° 42′, 37° 46′.
9. (i) 62° 4′; (ii) 69° 27′.
10. (i) 379·3 sq. ft.; (ii) 8·96 ft.; (iii) 33° 56′; (iv) 35° 32′, 63° 26′.
11. 35° 16′.
12. 1 in 10·64.
13. 14° 26′.
14. 8 sq. in.
15. 138 sq. in.
16. 7·3 sq. ft. (approx.).
17. 7·746 in.
18. 6,082 ft.
19. (i) 3,558 ml.; (ii) 2,662 ml.
20. 9,920 ml.

## EXERCISE XXIII (Page 189)

1. 29·53 lb.
2. 0·315 in., 0·630 in., 0·794 in.
3. 2·83; 22·63.
4. 6·30 cm.
5. 5·34 in.
6. 117 : 8.
7. 1·24 in.
8. 70·15 cu. in.

9. 0·276 in.
10. 8·8 cu. ft.
11. 60·3 cu. cm.; 110·9 sq. cm.
12. 9·43 in.
13. 2·03 in.
14. 4·91 lb.
15. 1·59 mm.
17. 208 ft. per min.
18. 2·68 ft., 1·34 ft.
19. $641\frac{3}{4}$ cu. ft.
20. 13,060 cu. ft. (approx.).
21. 1·35 cu. ft.
22. 77·7 sq. in.
23. 1·05 cu. in.
24. 369 cu. in.
25. 17·4 cu. in.
26. 52·6 cu. in.
27. 130·8 sq. in.
28. 1 : 0·806 : 1·201.
29. $\pi l D/d$.
30. 30·16 cu. cm.
31. 0·368 cu. in.
32. 9/32.
33. 16·93 cu. in.
34. 2·63 oz.

## EXERCISE XXIV (Page 205)

1. (i) 0·8218; (ii) 2·3926; (iii) 0·8800; (iv) 1·0258; (v) 0·1552, (vi) 1·8190; (vii) 0·6424; (viii) 1·0218.
2. (i) $a/c$; (ii) $a/b$; (iii) $c/b$; (iv) $a/b$; (v) $a/c$; (vi) $c/a$.
3. $\frac{\sqrt{3}}{2}$, 2, $\sqrt{3}$, $\sqrt{2}$, $\frac{2}{\sqrt{3}}$, $\frac{1}{\sqrt{3}}$.
4. (i) 1·754 in., 3·595 in.; (ii) 6·882 in., 8·506 in.; (iii) 3·057 cm., 10·457 cm.
5. 36·6 yd.
6. 1·09 in.
7. 22·48 in.
8. 13 ft. 3 in.
9. 39·6 sq. cm.; 5·66 cm.
10. (i) 33° 22′; (ii) 80° 41′; (iii) 66° 8′.
11. (i) 13° 21′; (ii) 73° 10′; (iii) 34° 51′.
12. 56° 15′.
13. (i) 38° 40′, 51° 20′; (ii) 38° 56′, 51° 4′; (iii) 25° 46′, 64° 14′.
15. 54° 3′
16. 181·2 sq. ft.
17. 10,072 cu. ft.
18. 16 ft. 9 in.; 6 ft. 7 in. (to the nearest inch).
19. 69° 54′, 7 ft. 6 in.
20. 9·334 in., 15·151 in.; 69·453 sq. in.
21. (i) 8·27 ml.; (ii) 39° 14′.
22. 16·35 cm.; 17·10 cm.
23. 920 watts.
24. 18·35 in., 27·05 in.
28. (i) 13 ft. $0\frac{3}{4}$ in.; (ii) 13 ft. $7\frac{1}{4}$ in.

## EXERCISE XXV (Page 213)

3. 65°.
6. 5/12.
7. $2\sqrt{3}/3$.
9. $\sqrt{2}$.
10. $\cot^2 \theta$.
11. sin A.
12. $\tan a$.
13. $\cos \theta$
14. 2 sec.$^2 x$.
15. 0.

## EXERCISE XXVI (Page 216)

1. 106° 16′; 3 in.
2. 70° 32′.
3. 22° 12′.
4. 4·36 in., 7·42 in., 20·34 sq. in.
5. 36°, 44° 28′.
6. (i) 41° 49′; (ii) 70° 32′.
7. 26° 34′, 19° 28′.
8. 1 (vertical) in 2·65 (along the path).
9. 32·66 sq. ft.

## EXERCISE XXVII (Page 220)

1. 6·01 ml.
2. 124 ft.
3. 10·76 ml.
4. 2,730 ft. (approx.), 176 m.p.h.
5. 132 ft.
6. 7 ft. 1 in.

## EXERCISE XXVIII (Page 221)

1. $2^\circ\ 52'$.
2. 11·95 in.
3. (i) $10^\circ\ 58'$; (ii) 10·44 ft.; (iii) 1·56 ft.
4. 2·184.
5. 4,122 cu. yd.
6. 11·91 ft.
7. $5^\circ\ 24'$.
8. $AE=BE=8{\cdot}83$ ft., $CE=DE=3{\cdot}97$ ft., $CD=2{\cdot}72$ ft.
9. $C\hat{A}D=C\hat{B}E=17^\circ\ 7'$, $A\hat{C}D=B\hat{C}E=20^\circ$, $A\hat{D}C=B\hat{E}C=142^\circ\ 53'$ $C\hat{D}E=C\hat{E}D=56^\circ\ 52'$, $D\hat{C}E=66^\circ\ 16'$.
10. 693 sq. ft.
11. S $11^\circ\ 43'$ W., S. $49^\circ\ 36'$ E.
12. 1 (vertical) in 7·77 (along the tunnel); 2,330 ft.
13. An equation; $0^\circ$, $90^\circ$.
14. $5{\cdot}34\times10^{-3}$.
15. $36^\circ\ 30'$.
16. $67^\circ\ 23'$.

## EXERCISE XXIX (Page 230)

1. +, −, −, +, +.
2. −, +, −, −, −.
3. $-\cos 72^\circ$, $+\sin 64^\circ$, $-\sin 10^\circ$, $+\tan 7^\circ\ 30'$.
4. $+\sin \pi/2$, $-\cos 68^\circ$, $-\tan 85^\circ\ 45'$, $-\cos \pi/4$.
5. $+\tan 80^\circ$, $+\sec 66^\circ\ 18'$, $+\sin(\pi-1{\cdot}82)=+\sin 1{\cdot}3216$, $-\operatorname{cosec} 34^\circ$.
6. 0·6428.
7. −0·7593.
8. −0·8480.
9. 0·7071.
10. −0·9573.
11. 0·2849.
12. −0·2790.
13. −0·3827.
14. −1·4142.
15. −0·1495.
16. 2·1301.
17. 1/2.
22. −230·1.
23. 5·5923.
24. (i) Second; (ii) fourth; (iii) third.
25. −0·6818.

## EXERCISE XXX (Page 245)

1. 0·7431; $45^\circ\ 14'$, $134^\circ\ 56'$.
2. (i) $90^\circ$, $\frac{1}{2}\pi^c$; (ii) $120^\circ$, $\frac{2}{3}\pi^c$; (iii) $36^\circ$, $\frac{1}{5}\pi^c$.
3. (i) $720^\circ$, $4\pi^c$; (ii) $240^\circ$, $\frac{4}{3}\pi^c$; (iii) $\dfrac{180^\circ}{p}$, $\dfrac{\pi^c}{p}$.
4. (i) 50; (ii) $\dfrac{20}{\pi}\simeq 6{\cdot}4$; (iii) 1.
5. (i) 5, $\frac{1}{2}\pi$, $\dfrac{2}{\pi}$; (ii) 230, $\frac{1}{25}$, 25.
6. (i) 3·6, $4\pi$, $\dfrac{1}{4\pi}$; (ii) 10, $\frac{1}{50}$, 50; (iii) 12, $\frac{1}{2}\pi$, $\dfrac{2}{\pi}$.
7. (i) $A$, $\dfrac{2\pi q}{p}$, $\dfrac{p}{2\pi q}$; (ii) $B$, $\dfrac{2\pi q}{p}$, $\dfrac{p}{2\pi q}$.
12. (i) 0·111 sec.; (ii) 3·26 ft. (or 99·4 cm.).
13. $y=3\sin \pi t$.

**14.** $y=3\sin\pi(t-1/4)$.

**15.** After $\dfrac{50-15\pi}{314} \simeq 0\cdot00916$ sec.; after $\dfrac{50-15\cdot5\pi}{314} \simeq 0\cdot00416$ sec.

**16.** (i) $\frac{3}{8}\pi$ radians; (ii) $\frac{3}{4}$ of a period; (iii) the first oscillation leads the second by $3\pi/32$.

**18.** $y=5\sin(3x+90°)$, *i.e.* $y=5\cos 3x$.

**19.** $x=4\sin(\frac{1}{2}\theta+22\frac{1}{2}°)$.

## EXERCISE XXXI (Page 251)

**1.** 8 in., 4 in. **9.** $\pi$. **10.** $2\pi$. **11.** $12\pi$.
**12.** 2. **13.** $2\pi$. **14.** $2\pi$. **15.** 50.
**16.** 15. **17.** 25.
**18.** *f*. If one of the frequencies is an integral multiple of the other, the resultant frequency is the smaller of the two.

## EXERCISE XXXII (Page 259)

**1.** 0, 1·11, 3·70. **2.** 0·86. **3.** 21·7, 141·7.
**4.** 0·4724 (radians), 21·98. **5.** 77°.
**6.** 1·935; 110° 51′. **7.** 10·9°. **8.** (i) First; (ii) third.
**9.** 19° 16′, 160° 44′. **10.** 61° 50′, 298° 10′. **11.** 246° 56′, 293° 4′.
**12.** 76° 42′, 256° 42′. **13.** 44° 25′, 315° 35′. **14.** 147° 40′, 212° 20′.
**15.** 54° 42′, −125° 18′. **16.** −15° 22′, −164° 38′.
**17.** $\theta=0\cdot2528$ or $2\cdot8888$. **18.** $x=\frac{3}{4}\pi$ or $\frac{7}{4}\pi$.
**19.** $a=1\cdot3695$ or $4\cdot9137$. **20.** $\theta=\frac{1}{6}\pi$ or $\frac{5}{6}\pi$.
**21.** 17°, 73°, 107°, 163°, 197°, 253°, 287°, 343°.
**22.** 118° 38′, 241° 22′. **23.** 105°, 165°, 285°, 345°.
**24.** 20° 36′, 80° 36′, 140° 36′, 200° 36′, 260° 36′, 320° 36′.
**25.** 0·077, 0·256, 0·744, 0·923 sec.
**26.** (i) After $\dfrac{50-15\pi}{314} \simeq 0\cdot00916$ sec.; (ii) after $\dfrac{50-15\cdot5\pi}{314} \simeq 0\cdot00416$ sec.
**27.** (i) 0·606; (ii) 0·955.
**28.** 125° 54′. **29.** 337° 23′. **30.** 208° 4′.

## EXERCISE XXXIII (Page 263)

**1.** 30°, 150°, 210°, 330°. **2.** 65° 54′, 114° 6′, 245° 54′, 294° 6′.
**3.** 35° 16′, 144° 44′, 215° 16′, 324° 44′. **4.** 270°.
**5.** 45°, 63° 26′, 225°, 243° 26′. **6.** 0°, 26° 34′, 180°, 206° 34′.
**7.** 101° 32′, 258°, 28′. **8.** 15°, 75°, 135°, 195°, 255°, 315°.
**9.** 0°, 180°, 360°. **10.** 51° 53′, 128° 7′, 231° 53′, 308° 7′.
**11.** 16° 9′. **12.** 7° 1′.

## EXERCISE XXXIV (Page 267)

**5.** $-\sin\theta$. **6.** $-\cos\theta$. **7.** $\sin\theta$.
**8.** $\tan\theta$. **9.** $-\cos\theta$. **10.** $-\cot\theta$.
**11.** 1. **12.** $\sin\theta$. **13.** $\tan^3 x$.
**19.** $x^2+y^2=2$. **20.** $26x^2-14xy+5y^2=81$. **21.** $4x^2-y^2=4$.

## EXERCISE XXXV (Page 281)

1. 11·79 lb. wt. at 47° 16′ with the force of 3 lb. wt.
2. 28·03 ft. per sec. at 15° 31′ with the velocity of 40 ft. per sec.
3. 9·434 tons wt. at 58° with the force of 5 tons wt.
4. 7·072 tons wt., N. 44° 25′ E.
5. 32·05 m.p.h., N. 4° 2′ W.
6. (i) $4_{115°}$; $-1·6904$, $3·6252$; (ii) $5_{210°}$; $-4·330$, $-2·5$; (iii) $2_{295°}$; $0·8152$, $-1·8126$.
7. $4·845_{268°\ 16'}$.
8. 23·3 m.p.h. at 9° 52′ with direction of ship's motion.
9. 141·9 m.p.h.; $15\frac{1}{2}$ sec.
10. $3·44_{153°\ 35'}$.
11. 41° 24′ with the bank.
12. $P \cos \theta - W \sin \alpha$.
13. 5·98 tons wt. E., 4·84 tons wt. N.E.
14. 37·74 ft. per sec. at 58° with the velocity of 20 ft. per sec.
15. 6·33 ml., N. 77° 23′ E.
16. $15·06_{105°\ 47'}$.
19. 62° 43′ between the first and second force, 153° 37′ between the second and third, 143° 40′ between the third and first.
20. $r=13$, $\theta=67° \ 23'$.
21. $r=5$, $\theta=-36° \ 52'$.
22. $r=2·865$, $\theta=29° \ 15'$.
23. $r=3·138$, $\theta=120° \ 39'$.
24. $15·06_{105°\ 47'}$.
25. 122·4 lb. wt.
26. P = 6·534, Q = 4·750.
27. See answers to Question 26.
30. $4·21_{63°\ 44'}$.
31. 11·72 lb. wt. in direction S. 38° 42′ E.

## EXERCISE XXXVI (Page 290)

9. $4·33 \sin 300t + 2·5 \cos 300t$; 1·191.
10. $15·92 \sin 100\pi t - 12·11 \cos 100\pi t$.
12. 63/65.
13. 24/25, −7/25, −24/7.
14. sin 70°.
15. cos 60°.
16. $\cos \frac{1}{2}\pi$.
17. sin (−30°), *i.e.* −sin 30°.
18. $\sin 3\theta$.
19. $\cos 2\alpha$.
20. cos 30°.
21. $\frac{1}{2} \sin x$.
24. $\dfrac{1+\sqrt{3}}{2\sqrt{2}} \simeq 0·9659$.
25. $\dfrac{1+\sqrt{3}}{2\sqrt{2}} \simeq 0·9659$.
26. $\frac{1}{2}\sqrt{2+\sqrt{2}} \simeq 0·9239$.
38. 30°, 150°, 270°.
39. 0°, 60°, 180°, 300°.
40. 0°, 52° 15′, 127° 45′, 180°, 232° 15′, 307° 45′.
41. 133° 11′, 313° 11′.
42. 90°, 180°.
43. 123° 24′, 236° 36′.
44. 56° 43′, 236° 43′.

## EXERCISE XXXVII (Page 297)

1. $8·062 \sin (\theta + 29° \ 45')$.
2. $3·176 \sin (\theta + 61° \ 10')$.
3. $13 \sin (\theta + 292° \ 37')$ *or* $13 \sin (\theta - 67° \ 23')$.
4. $129·3 \sin (\theta + 320° \ 39')$ *or* $129·3 \sin (\theta - 39° \ 21')$.

5. $5{\cdot}007 \sin(\theta+67° 19')$.
6. $27{\cdot}31 \sin(300t-1{\cdot}1563)$.
7. $\pm\sqrt{a^2+b^2}$.
8. $\pm 5$; 143° 8′, 323° 8′.
9. $1{\cdot}628 \sin(2\pi ft+0{\cdot}1853)$.
10. $2 \sin(\theta+\frac{1}{3}\pi)$.
11. $17 \sin(60\pi t+0{\cdot}4899)$; $\frac{1}{30}$; 17, 8.
12. 104° 24′, 330° 20′.
13. 19° 50′, 160° 10′.
14. 249° 38′.
15. 0°, 126° 52′, 360°.
16. 7° 42′, 138° 58′.
17. 10·549; 73° 20′.
18. $13{\cdot}98 \sin(3wt-0{\cdot}6536)$; $0{\cdot}0081 \pm n\times 0{\cdot}0349$ *or* $0{\cdot}0167 \pm n \times 0{\cdot}0349$.
19. 0° to 126° 52′, 306° 52′ to 360°.
20. 12° 3′.
21. $Z=\sqrt{R^2+L^2\omega^2}$, $\phi=\tan^{-1}(L\omega/R)$; $Z=12{\cdot}36$, $\phi=0{\cdot}867$.
22. (i) $A+B$, $A-B$; (ii) $c+\sqrt{a^2+b^2}$, $c-\sqrt{a^2+b^2}$; (iii) $A+\sqrt{B^2+C^2}$, $A-\sqrt{B^2+C^2}$.

## EXERCISE XXXVIII (Page 302)

4. 850,000 miles (approx.).
5. 3·8 miles (approx.).
6. 0·9949.
7. 0·00022 in.
8. 0·719.
9. 0·71934.

## EXERCISE XXXIX (Page 302)

4. 0·5477, 0·8367, 0·6546.
6. ±79° 16′, ±100° 44′.
8. $a=-2\sqrt{3}$, $b=1$; $3{\cdot}606 \sin(\omega t+163° 54')$.
9. $a=\sqrt{3}$, $b=1$.
11. 2·153, 7·486.

## EXERCISE XL (Page 317)

1. (i) 6·792 in.; (ii) 6·594 cm.
2. 10·64 cm.
3. 2·901 in.
4. 28° 32′.
5. 21·60 in.
6. 120°.
7. 15·10 cm.
8. 118 sq. ft.
9. 6·724 sq. in.
10. 5·764 in.; 20·66 sq. in.
11. 2 ml. 1,690 yd.
12. 8 ft. 0 in.
13. 7·17 ft.
14. 144° 7′.
15. 65·03 ml.
16. $BC=6{\cdot}93$ ft., $AC=8$ ft., $CD=5{\cdot}04$ ft.
17. 9·102 lb. wt.
18. 67·2 m.p.h. from direction S. 71° 3′ E.
19. 4·71 tons wt. at an angle 86° 28′ with the force of 2 tons wt.
20. 7·926 in.
21. 78·25 sq. cm.
22. 18·23 in.; 24·63 sq. in.
23. 0·291 sq. in.
24. 3·812 cm.
25. 8·67 sq. in.
26. 19,950 cu. yd. (approx.).
27. 22·93 sq. in.
28. See answer to Question 27.

## EXERCISE XLI (Page 330)

1. $A=81° 29'$, $B=62° 19'$, $C=36° 12'$.
2. $A=20° 24'$, $B=118° 14'$, $C=41° 22'$.
3. $B=74°$, $a=1{\cdot}654$ in., $c=3$ in.
4. $a=14{\cdot}13$ ft., $b=20{\cdot}61$ ft., $A=29°$.

5. $c=3\cdot158$ cm., $A=39°\ 6'$, $B=80°\ 54'$.
6. $b=29$ ml. (approx.), $A=136°\ 24'$, $C=32°\ 11'$.
7. $a=3\cdot985$ in., $A=63°\ 46'$, $C=64°\ 14'$, *or* $a=0\cdot941$ in., $A=12°\ 14'$, $C=115°\ 46'$.
8. $c=4\cdot550$ cm., $B=54°\ 30'$, $C=47°\ 48'$.
9. $b=1\cdot164$ ft., $A=20°\ 42'$, $B=24°\ 18'$.
10. (i) 10 cm., 5·177 cm., 75° ; (ii) 18° 11′, 45° 13′, 116° 36′ ; (iii) 4·022 in., 27° 33′, 68° 27′ ; (iv) 183·5 yd., 29° 3′, 33° 57′.
11. 64·82 ft.
12. 0·317 sq. in.
13. 77° 10′
14. 33° 41′ *or* 18° 26′.
15. $DF=18\cdot70$ yd., $EF=6\cdot18$ yd.
16. 1·52 ft.
17. 1·45 ft. ; 0·62 ft.
18. 6·59 ft., 7·78 ft.

## EXERCISE XLII (Page 336)

1. 2·945 ml.
2. 3 in.
3. 43·8 ft.
4. 66° 56′.
5. 66·62 ft.
6. 4·68 ft., 3·52 ft., 4·68 ft.
7. 379 yd.
9. 15·7 knots, N. 66° 38′ E.

## EXERCISE XLIII (Page 338)

1. 8 ft. 5 in.
2. 12·37 knots.
3. 24·93 ft.
4. By car.
5. 135 yd.
6. $AD=7'\ 10''$, $BD=11'\ 3''$.
7. 1,075 gall. (approx.).
8. 32·1 nautical miles ; S. 22° 36′ E.
9. 43·2 yd.
10. 183 m.p.h.
11. $A=78°\ 36'$, $C=71°\ 24'$, $b=2\cdot244$ in., $c=4\cdot256$ in.
12. 15·78 in.

## EXERCISE XLIV (Page 351)

1. (*a*) 300 m.p.h. ; (b) 150 m.p.h. ; (*c*) 180 m.p.h.
2. 8·5° C. per min.
3. $n=40-\frac{t}{15}$ ; $-\frac{1}{15}$ gal. per min.
4. Approx. 10,000 ft./sec.
5. 2·225 in. per year : (*a*) $4\frac{1}{2}$ to $8\frac{1}{2}$ yr. ; (*b*) 6 to 7 yr. ; (*c*) 5 in. per yr.
6. 22 ft./sec., $-0\cdot6$ ft./sec.$^2$.
7. $\frac{\delta x}{\delta t}$ in. per yr. ; $\frac{dx}{dt}$ in. per yr.
8. After 50 sec. : (*a*) after about 30 sec. ; (*b*) after about 65 sec.
9. 5.
10. Negative.
11. 0·045, 0·055.
12. (*a*) $t=2$ to 3 ; (*b*) $t=9$ to 11.
13. $1\frac{3}{4}$ sec., $3\frac{1}{4}$ sec. ; body is falling.

## EXERCISE XLV (Page 357)

1. (*a*) 1·6 ; (*b*) 4·6.
3. 0·016, 0·026, 0·02.
5. $4\pi$, $9\pi$.
6. 0·22.
7. 8·192, 8·624, 1080.
8. 0·9657, 1·0355, 0·0349.

## EXERCISE XLVI (Page 366)

1. (a) 48 ft./sec., 112 ft./sec.; (b) 97·6 ft./sec., 96·16 ft./sec.; (c) 96·0016 ft./sec.; (d) 96·000016 ft./sec.
2. $(32t+0\cdot16)$ ft./sec., $32t+0\cdot0016$ ft./sec., $32t$ ft./sec.
3. $12x+6h$, $12x$.
4. $12x\delta x+6\delta x^2$, $12x+6\delta x$, $12x$.
5. $-\dfrac{\delta x}{x(x+\delta x)}$, $-\dfrac{1}{x^2}$.
6. $3x^2h+3xh^2+h^3$, $3x^2+3xh+h^2$, $3x^2$.
7. $\dfrac{dy}{dx}=a$.
8. $\dfrac{2}{x^3}$.
9. $9x^2$.
10. $-4x$.
11. $720x^7$.
12. $8x^3-2x^5$.
13. $12x^2-14x+21$.
14. $0\cdot02-5\cdot08x-2.16x^2$.
15. $-3/x^4$.
16. $-1/2x^5$.
17. $4\cdot8^{0\cdot6}$.
18. $-\dfrac{1}{2x^3}+\dfrac{1}{2x^2}$.
19. $10t$.
20. $\frac{3}{5}x^2$.
21. $0\cdot2r$.
22. $8000x^3$.
23. $49t^6$.
24. $8p+20p^3$.
25. $x^{n-1}$.
26. $u+gt$.
27. $-100/v^2$.
28. $-k/r^{k+1}$.
29. $9\cdot6\times10^{-6}V^{3\cdot8}$.
30. $2/\sqrt{z}$.
31. $-\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{4}$.
32. 16 ft./sec. upwards, 16 ft./sec. downwards, after $2\frac{1}{2}$ sec., 100 ft.
33. $-1$, 1.
35. (0, 6), (2, 2).

## EXERCISE XLVII (Page 370)

1. Max. 1.
2. Max. 0; min. $-4$.
3. Max. $1\frac{1}{4}$.
4. Min. $-9$.
5. Min. 0.
6. Max. 2; min. $-2$.
7. $wl^2/8$.
8. A cube of side $\sqrt[3]{6\cdot4}$ ft. $\simeq 1\cdot86$ ft.
9. 64.
10. $\frac{3}{8}$ amps.
11. Least 3; greatest 4.
12. $9Wl^2/128$.

## EXERCISE XLVIII (Page 375)

1. 64 ft.
2. 660 yd.
3. 550 rev.
4. 5 ml., $\frac{44}{135}$ ft./sec.$^2$.

## EXERCISE XLIX (Page 382)

1. 26 sec.
3. 0·023.
8. 3·2.
9. $x^2+c$, $b^2-a^2$.
10. $\frac{1}{6}t^3+c$, $2\frac{1}{3}$.
11. 240 ft.
12. $3x^2+c$.
13. $3x^2+5x+c$.
14. $3x^3+c$.
15. $3x^3+3x^2+5x+c$.
16. $\frac{5}{6}x^6+c$.
17. $-\dfrac{1}{x}+c$.
18. $x^{1\cdot8}+c$.
19. $3x^2-\dfrac{1}{x}+c$

## EXERCISE L (Page 387)

1. $\frac{1}{8}x^8+c$.
2. $2x^5+c$.
3. $\frac{1}{10}v^5+c$.
4. $0{\cdot}01t^{10}+c$.
5. $\frac{2}{5}\sqrt{r^5}+c$.
6. $-1/2x+c$.
7. $-1/3m^3+c$.
8. $\frac{10}{3}p^{2\cdot6}+c$.
9. 8.
10. 16.
11. $1\frac{1}{4}$.
12. $\frac{4}{3}\pi a^3$.
13. $\frac{31}{160}$.
14. $\frac{1}{6}$.
15. $100\left(1-\frac{1}{\sqrt[5]{2}}\right)$.
16. $\frac{8}{3}(2-\sqrt{2})$.
17. $\frac{2a^7}{7}$.
18. $\frac{1}{n-1}\left(\frac{1}{v_1^{n-1}}-\frac{1}{v_2^{n-1}}\right)$
19. $\frac{5}{18}$.
20. $x-\frac{1}{3}x^3+c$.
21. $v^2-v^4+c$.
22. $\frac{1}{4}x^4-\frac{1}{2}x^2+c$.
23. 36.
24. $\frac{1}{n+1}(b^{n+1}-a^{n+1})$.
25. $\frac{28}{3}$.
26. $\frac{1}{2}(b^2-a^2)$.
27. 48 ft.
28. $(2t^2+6)$ ft./sec., 56 ft./sec.
29. $y=\frac{1}{3}x^3+c$, where $c$ is the value of $y$ at $x=o$.
30. $x=2$ and $x=3$, $\frac{1}{6}$.

## EXERCISE LI (Page 395)

1. 5 acres.
2. (*a*) 1,863,000 gal. ; (*b*) 3,117,000 gal.
3. 29·1 cu. ft.
4. 460 in. lb. wt., 153 lb. wt.
5. 7960 ft. lb. ; 31·8 lb. wt./ft.$^2$.
6. 62310 cu. ft., 316 sq. ft.
7. Increase in velocity from $t=a$ to $t=b$ ; 27·3 ft./sec.
12. 154 ft. lb. wt., 308 lb. wt./ft.$^2$.
13. $\frac{1}{2}x$ lb. wt., 25 ft. lb. wt.
15. $\frac{1}{3}a^2$.
16. 2.
17. $\frac{1}{2}\pi$.
18. $64\pi$ cu. in.
20. $\frac{152\pi}{3}$ cu. in.

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| 45 | [illegible] | [illegible] | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 46 | [illegible] | [illegible] | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | [illegible] | [illegible] | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 48 | 6812 | 6821 | 6830 | 6839 | 6848 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 49 | 6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 50 | [illegible] | [illegible] | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
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| **62** | 7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| **63** | 7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **64** | 8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **65** | 8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **66** | 8195 | 8202 | 8209 | 8215 | 8222 | 8228 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **67** | 8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 8319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **68** | 8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| **69** | 8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| **70** | 8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| **71** | 8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **72** | 8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **73** | 8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **74** | 8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **75** | 8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **76** | 8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **77** | 8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **78** | 8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **79** | 8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **80** | 9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **81** | 9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **82** | 9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **83** | 9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **84** | 9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9284 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **85** | 9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **86** | 9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **87** | 9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **88** | 9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **89** | 9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **90** | 9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **91** | 9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **92** | 9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **93** | 9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **94** | 9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **95** | 9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **96** | 9823 | 9827 | 9832 | 9836 | 9841 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **97** | 9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **98** | 9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **99** | 9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |

# ANTI-LOGARITHMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Differences 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ·00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| ·04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| ·07 | 1175 | 1178 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | [illegible] | 2 | 2 | 2 | 2 |
| ·08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| ·11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| ·13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·15 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·17 | 1479 | 1483 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·18 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| ·19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| ·21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1687 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| ·23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| ·26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| ·28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| ·33 | [illegible] | [illegible] | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 34 | [illegible] | [illegible] | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2286 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| ·39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| ·40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| ·42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| ·43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| ·45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| ·48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| ·49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |

# ANTI-LOGARITHMS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·50 | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| ·51 | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·52 | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| ·53 | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·54 | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| ·55 | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| ·56 | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·57 | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| ·58 | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| ·59 | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| ·60 | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| ·61 | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·62 | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·63 | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·64 | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·65 | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| ·66 | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| ·67 | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| ·68 | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| ·69 | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| ·70 | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| ·71 | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| ·72 | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| ·73 | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| ·74 | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5585 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| ·75 | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| ·76 | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5848 | 5861 | 5875 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| ·77 | 5888 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5984 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| ·78 | 6026 | 6039 | 6053 | 6067 | 6081 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| ·79 | 6166 | 6180 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| ·80 | 6310 | 6324 | 6339 | 6353 | 6368 | 6383 | 6397 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| ·81 | [illegible] | [illegible] | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·82 | [illegible] | [illegible] | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| ·83 | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| ·84 | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 15 |
| ·85 | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7228 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·86 | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| ·87 | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| ·88 | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| ·89 | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| ·90 | [illegible] | [illegible] | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 16 | 17 |
| ·91 | [illegible] | [illegible] | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| ·92 | [illegible] | [illegible] | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| ·93 | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·94 | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| ·95 | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 12 | 15 | 17 | 19 |
| ·96 | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| ·97 | [illegible] | [illegible] | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| ·98 | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| ·99 | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 20 |

# NATURAL SINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ·0000 | 0017 | 0035 | 0052 | 0070 | 0087 | 0105 | 0122 | 0140 | 0157 | 3 | 6 | 9 | 12 | 15 |
| 1 | ·0175 | 0192 | 0209 | 0227 | 0244 | 0262 | 0279 | 0297 | 0314 | 0332 | 3 | 6 | 9 | 12 | 15 |
| 2 | ·0349 | 0366 | 0384 | 0401 | 0419 | 0436 | 0454 | 0471 | 0488 | 0506 | 3 | 6 | 9 | 12 | 15 |
| 3 | ·0523 | 0541 | 0558 | 0576 | 0593 | 0610 | 0628 | 0645 | 0663 | 0680 | 3 | 6 | 9 | 12 | 15 |
| 4 | ·0698 | 0715 | 0732 | 0750 | 0767 | 0785 | 0802 | 0819 | 0837 | 0854 | 3 | 6 | 9 | 12 | 14 |
| 5 | ·0872 | 0889 | 0906 | 0924 | 0941 | 0958 | 0976 | 0993 | 1011 | 1028 | 3 | 6 | 9 | 12 | 14 |
| 6 | ·1045 | 1063 | 1080 | 1097 | 1115 | 1132 | 1149 | 1167 | 1184 | 1201 | 3 | 6 | 9 | 12 | 14 |
| 7 | ·1219 | 1236 | 1253 | 1271 | 1288 | 1305 | 1323 | 1340 | 1357 | 1374 | 3 | 6 | 9 | 12 | 14 |
| 8 | ·1392 | 1409 | 1426 | 1444 | 1461 | 1478 | 1495 | 1513 | 1530 | 1547 | 3 | 6 | 9 | 12 | 14 |
| 9 | ·1564 | 1582 | 1599 | 1616 | 1633 | 1650 | 1668 | 1685 | 1702 | 1719 | 3 | 6 | 9 | 12 | 14 |
| 10 | ·1736 | 1754 | 1771 | 1788 | 1805 | 1822 | 1840 | 1857 | 1874 | 1891 | 3 | 6 | 9 | 11 | 14 |
| 11 | ·1908 | 1925 | 1942 | 1959 | 1977 | 1994 | 2011 | 2028 | 2045 | 2062 | 3 | 6 | 9 | 11 | 14 |
| 12 | ·2079 | 2096 | 2113 | 2130 | 2147 | 2164 | 2181 | 2198 | 2215 | 2233 | 3 | 6 | 9 | 11 | 14 |
| 13 | ·2250 | 2267 | 2284 | 2300 | 2317 | 2334 | 2351 | 2368 | 2385 | 2402 | 3 | 6 | 8 | 11 | 14 |
| 14 | ·2419 | 2436 | 2453 | 2470 | 2487 | 2504 | 2521 | 2538 | 2554 | 2571 | 3 | 6 | 8 | 11 | 14 |
| 15 | ·2588 | 2605 | 2622 | 2639 | 2656 | 2672 | 2689 | 2706 | 2723 | 2740 | 3 | 6 | 8 | 11 | 14 |
| 16 | ·2756 | 2773 | 2790 | 2807 | 2823 | 2840 | 2857 | 2874 | 2890 | 2907 | 3 | 6 | 8 | 11 | 14 |
| 17 | ·2924 | 2940 | 2957 | 2974 | 2990 | 3007 | 3024 | 3040 | 3057 | 3074 | 3 | 6 | 8 | 11 | 14 |
| 18 | ·3090 | 3107 | 3123 | 3140 | 3156 | 3173 | 3190 | 3206 | 3223 | 3239 | 3 | 6 | 8 | 11 | 14 |
| 19 | ·3256 | 3272 | 3289 | 3305 | 3322 | 3338 | 3355 | 3371 | 3387 | 3404 | 3 | 5 | 8 | 11 | 14 |
| 20 | ·3420 | 3437 | 3453 | 3469 | 3486 | 3502 | 3518 | 3535 | 3551 | 3567 | 3 | 5 | 8 | 11 | 14 |
| 21 | ·3584 | 3600 | 3616 | 3633 | 3649 | 3665 | 3681 | 3697 | 3714 | 3730 | 3 | 5 | 8 | 11 | 14 |
| 22 | ·3746 | 3762 | 3778 | 3795 | 3811 | 3827 | 3843 | 3859 | 3875 | 3891 | 3 | 5 | 8 | 11 | 14 |
| 23 | ·3907 | 3923 | 3939 | 3955 | 3971 | 3987 | 4003 | 4019 | 4035 | 4051 | 3 | 5 | 8 | 11 | 14 |
| 24 | ·4067 | 4083 | 4099 | 4115 | 4131 | 4147 | 4163 | 4179 | 4195 | 4210 | 3 | 5 | 8 | 11 | 13 |
| 25 | ·4226 | 4242 | 4258 | 4274 | 4289 | 4305 | 4321 | 4337 | 4352 | 4368 | 3 | 5 | 8 | 11 | 13 |
| 26 | ·4384 | 4399 | 4415 | 4431 | 4446 | 4462 | 4478 | 4493 | 4509 | 4524 | 3 | 5 | 8 | 10 | 13 |
| 27 | ·4540 | 4555 | 4571 | 4586 | 4602 | 4617 | 4633 | 4648 | 4664 | 4679 | 3 | 5 | 8 | 10 | 13 |
| 28 | ·4695 | 4710 | 4726 | 4741 | 4756 | 4772 | 4787 | 4802 | 4818 | 4833 | 3 | 5 | 8 | 10 | 13 |
| 29 | ·4848 | 4863 | 4879 | 4894 | 4909 | 4924 | 4939 | 4955 | 4970 | 4985 | 3 | 5 | 8 | 10 | 13 |
| 30 | ·5000 | 5015 | 5030 | 5045 | 5060 | 5075 | 5090 | 5105 | 5120 | 5135 | 3 | 5 | 8 | 10 | 13 |
| 31 | ·5150 | 5165 | 5180 | 5195 | 5210 | 5225 | 5240 | 5255 | 5270 | 5284 | 2 | 5 | 7 | 10 | 12 |
| 32 | ·5299 | 5314 | 5329 | 5344 | 5358 | 5373 | 5388 | 5402 | 5417 | 5432 | 2 | 5 | 7 | 10 | 12 |
| 33 | ·5446 | 5461 | 5476 | 5490 | 5505 | 5519 | 5534 | 5548 | 5563 | 5577 | 2 | 5 | 7 | 10 | 12 |
| 34 | ·5592 | 5606 | 5621 | 5635 | 5650 | 5664 | 5678 | 5693 | 5707 | 5721 | 2 | 5 | 7 | 10 | 12 |
| 35 | ·5736 | 5750 | 5764 | 5779 | 5793 | 5807 | 5821 | 5835 | 5850 | 5864 | 2 | 5 | 7 | 9 | 12 |
| 36 | ·5878 | 5892 | 5906 | 5920 | 5934 | 5948 | 5962 | 5976 | 5990 | 6004 | 2 | 5 | 7 | 9 | 12 |
| 37 | ·6018 | 6032 | 6046 | 6060 | 6074 | 6088 | 6101 | 6115 | 6129 | 6143 | 2 | 5 | 7 | 9 | 12 |
| 38 | ·6157 | 6170 | 6184 | 6198 | 6211 | 6225 | 6239 | 6252 | 6266 | 6280 | 2 | 5 | 7 | 9 | 11 |
| 39 | ·6293 | 6307 | 6320 | 6334 | 6347 | 6361 | 6374 | 6388 | 6401 | 6414 | 2 | 4 | 7 | 9 | 11 |
| 40 | ·6428 | 6441 | 6455 | 6468 | 6481 | 6494 | 6508 | 6521 | 6534 | 6547 | 2 | 4 | 7 | 9 | 11 |
| 41 | ·6561 | 6574 | 6587 | 6600 | 6613 | 6626 | 6639 | 6652 | 6665 | 6678 | 2 | 4 | 7 | 9 | 11 |
| 42 | ·6691 | 6704 | 6717 | 6730 | 6743 | 6756 | 6769 | 6782 | 6794 | 6807 | 2 | 4 | 6 | 9 | 11 |
| 43 | ·6820 | 6833 | 6845 | 6858 | 6871 | 6884 | 6896 | 6909 | 6921 | 6934 | 2 | 4 | 6 | 8 | 11 |
| 44 | ·6947 | 6959 | 6972 | 6984 | 6997 | 7009 | 7022 | 7034 | 7046 | 7059 | 2 | 4 | 6 | 8 | 10 |

# NATURAL SINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | ·7071 | 7083 | 7096 | 7108 | 7120 | 7133 | 7145 | 7157 | 7169 | 7181 | 2 | 4 | 6 | 8 | 10 |
| 46 | ·7193 | 7206 | 7218 | 7230 | 7242 | 7254 | 7266 | 7278 | 7290 | 7302 | 2 | 4 | 6 | 8 | 10 |
| 47 | ·7314 | 7325 | 7337 | 7349 | 7361 | 7373 | 7385 | 7396 | 7408 | 7420 | 2 | 4 | 6 | 8 | 10 |
| 48 | ·7431 | 7443 | 7455 | 7466 | 7478 | 7490 | 7501 | 7513 | 7524 | 7536 | 2 | 4 | 6 | 8 | 10 |
| 49 | ·7547 | 7559 | 7570 | 7581 | 7593 | 7604 | 7615 | 7627 | 7638 | 7649 | 2 | 4 | 6 | 8 | 9 |
| 50 | ·7660 | 7672 | 7683 | 7694 | 7705 | 7716 | 7727 | 7738 | 7749 | 7760 | 2 | 4 | 6 | 7 | 9 |
| 51 | ·7771 | 7782 | 7793 | 7804 | 7815 | 7826 | 7837 | 7848 | 7859 | 7869 | 2 | 4 | 5 | 7 | 9 |
| 52 | ·7880 | 7891 | 7902 | 7912 | 7923 | 7934 | 7944 | 7955 | 7965 | 7976 | 2 | 4 | 5 | 7 | 9 |
| 53 | ·7986 | 7997 | 8007 | 8018 | 8028 | 8039 | 8049 | 8059 | 8070 | 8080 | 2 | 3 | 5 | 7 | 9 |
| 54 | ·8090 | 8100 | 8111 | 8121 | 8131 | 8141 | 8151 | 8161 | 8171 | 8181 | 2 | 3 | 5 | 7 | 8 |
| 55 | ·8192 | 8202 | 8211 | 8221 | 8231 | 8241 | 8251 | 8261 | 8271 | 8281 | 2 | 3 | 5 | 7 | 8 |
| 56 | ·8290 | 8300 | 8310 | 8320 | 8329 | 8339 | 8348 | 8358 | 8368 | 8377 | 2 | 3 | 5 | 6 | 8 |
| 57 | ·8387 | 8396 | 8406 | 8415 | 8425 | 8434 | 8443 | 8453 | 8462 | 8471 | 2 | 3 | 5 | 6 | 8 |
| 58 | ·8480 | 8490 | 8499 | 8508 | 8517 | 8526 | 8536 | 8545 | 8554 | 8563 | 2 | 3 | 5 | 6 | 8 |
| 59 | ·8572 | 8581 | 8590 | 8599 | 8607 | 8616 | 8625 | 8634 | 8643 | 8652 | 1 | 3 | 4 | 6 | 7 |
| 60 | ·8660 | 8669 | 8678 | 8686 | 8695 | 8704 | 8712 | 8721 | 8729 | 8738 | 1 | 3 | 4 | 6 | 7 |
| 61 | ·8746 | 8755 | 8763 | 8771 | 8780 | 8788 | 8796 | 8805 | 8813 | 8821 | 1 | 3 | 4 | 6 | 7 |
| 62 | ·8829 | 8838 | 8846 | 8854 | 8862 | 8870 | 8878 | 8886 | 8894 | 8902 | 1 | 3 | 4 | 5 | 7 |
| 63 | ·8910 | 8918 | 8926 | 8934 | 8942 | 8949 | 8957 | 8965 | 8973 | 8980 | 1 | 3 | 4 | 5 | 6 |
| 64 | ·8988 | 8996 | 9003 | 9011 | 9018 | 9026 | 9033 | 9041 | 9048 | 9056 | 1 | 3 | 4 | 5 | 6 |
| 65 | ·9063 | 9070 | 9078 | 9085 | 9092 | 9100 | 9107 | 9114 | 9121 | 9128 | 1 | 2 | 4 | 5 | 6 |
| 66 | ·9135 | 9143 | 9150 | 9157 | 9164 | 9171 | 9178 | 9184 | 9191 | 9198 | 1 | 2 | 3 | 5 | 6 |
| 67 | ·9205 | 9212 | 9219 | 9225 | 9232 | 9239 | 9245 | 9252 | 9259 | 9265 | 1 | 2 | 3 | 4 | 6 |
| 68 | ·9272 | 9278 | 9285 | 9291 | 9298 | 9304 | 9311 | 9317 | 9323 | 9330 | 1 | 2 | 3 | 4 | 5 |
| 69 | ·9336 | 9342 | 9348 | 9354 | 9361 | 9367 | 9373 | 9379 | 9385 | 9391 | 1 | 2 | 3 | 4 | 5 |
| 70 | ·9397 | 9403 | 9409 | 9415 | 9421 | 9426 | 9432 | 9438 | 9444 | 9449 | 1 | 2 | 3 | 4 | 5 |
| 71 | ·9455 | 9461 | 9466 | 9472 | 9478 | 9483 | 9489 | 9494 | 9500 | 9505 | 1 | 2 | 3 | 4 | 5 |
| 72 | ·9511 | 9516 | 9521 | 9527 | 9532 | 9537 | 9542 | 9548 | 9553 | 9558 | 1 | 2 | 3 | 3 | 4 |
| 73 | ·9563 | 9568 | 9573 | 9578 | 9583 | 9588 | 9593 | 9598 | 9603 | 9608 | 1 | 2 | 2 | 3 | 4 |
| 74 | ·9613 | 9617 | 9622 | 9627 | 9632 | 9636 | 9641 | 9646 | 9650 | 9655 | 1 | 2 | 2 | 3 | 4 |
| 75 | ·9659 | 9664 | 9668 | 9673 | 9677 | 9681 | 9686 | 9690 | 9694 | 9699 | 1 | 1 | 2 | 3 | 4 |
| 76 | ·9703 | 9707 | 9711 | 9715 | 9720 | 9724 | 9728 | 9732 | 9736 | 9740 | 1 | 1 | 2 | 3 | 3 |
| 77 | ·9744 | 9748 | 9751 | 9755 | 9759 | 9763 | 9767 | 9770 | 9774 | 9778 | 1 | 1 | 2 | 3 | 3 |
| 78 | ·9781 | 9785 | 9789 | 9792 | 9796 | 9799 | 9803 | 9806 | 9810 | 9813 | 1 | 1 | 2 | 2 | 3 |
| 79 | ·9816 | 9820 | 9823 | 9826 | 9829 | 9833 | 9836 | 9839 | 9842 | 9845 | 1 | 1 | 2 | 2 | 3 |
| 80 | ·9848 | 9851 | 9854 | 9857 | 9860 | 9863 | 9866 | 9869 | 9871 | 9874 | 0 | 1 | 1 | 2 | 2 |
| 81 | ·9877 | 9880 | 9882 | 9885 | 9888 | 9890 | 9893 | 9895 | 9898 | 9900 | 0 | 1 | 1 | 2 | 2 |
| 82 | ·9903 | 9905 | 9907 | 9910 | 9912 | 9914 | 9917 | 9919 | 9921 | 9923 | 0 | 1 | 1 | 2 | 2 |
| 83 | ·9925 | 9928 | 9930 | 9932 | 9934 | 9936 | 9938 | 9940 | 9942 | 9943 | 0 | 1 | 1 | 1 | 2 |
| 84 | ·9945 | 9947 | 9949 | 9951 | 9952 | 9954 | 9956 | 9957 | 9959 | 9960 | 0 | 1 | 1 | 1 | 1 |
| 85 | ·9962 | 9963 | 9965 | 9966 | 9968 | 9969 | 9971 | 9972 | 9973 | 9974 | 0 | 0 | 1 | 1 | 1 |
| 86 | ·9976 | 9977 | 9978 | 9979 | 9980 | 9981 | 9982 | 9983 | 9984 | 9985 | 0 | 0 | 1 | 1 | 1 |
| 87 | ·9986 | 9987 | 9988 | 9989 | 9990 | 9990 | 9991 | 9992 | 9993 | 9993 | | | | | |
| 88 | ·9994 | 9995 | 9995 | 9996 | 9996 | 9997 | 9997 | 9997 | 9998 | 9998 | | | | | |
| 89 | ·9998 | 9999 | 9999 | 9999 | 9999 | 1·000 | 1·000 | 1·000 | 1·000 | 1·000 | | | | | |

# NATURAL COSINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Subtract Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1·000 | 1·000 | 1·000 | 1·000 | 1·000 | 1·000 | 9999 | 9999 | 9999 | 9999 | | | | | |
| 1 | ·9998 | 9998 | 9998 | 9997 | 9997 | 9997 | 9996 | 9996 | 9995 | 9995 | | | | | |
| 2 | ·9994 | 9993 | 9993 | 9992 | 9991 | 9990 | 9990 | 9989 | 9988 | 9987 | | | | | |
| 3 | ·9986 | 9985 | 9984 | 9983 | 9982 | 9981 | 9980 | 9979 | 9978 | 9977 | 0 | 0 | 1 | 1 | 1 |
| 4 | ·9976 | 9974 | 9973 | 9972 | 9971 | 9969 | 9968 | 9966 | 9965 | 9963 | 0 | 0 | 1 | 1 | 1 |
| 5 | ·9962 | 9960 | 9959 | 9957 | 9956 | 9954 | 9952 | 9951 | 9949 | 9947 | 0 | 1 | 1 | 1 | 1 |
| 6 | ·9945 | 9943 | 9942 | 9940 | 9938 | 9936 | 9934 | 9932 | 9930 | 9928 | 0 | 1 | 1 | 1 | 2 |
| 7 | ·9925 | 9923 | 9921 | 9919 | 9917 | 9914 | 9912 | 9910 | 9907 | 9905 | 0 | 1 | 1 | 2 | 2 |
| 8 | ·9903 | 9900 | 9898 | 9895 | 9893 | 9890 | 9888 | 9885 | 9882 | 9880 | 0 | 1 | 1 | 2 | 2 |
| 9 | ·9877 | 9874 | 9871 | 9869 | 9866 | 9863 | 9860 | 9857 | 9854 | 9851 | 0 | 1 | 1 | 2 | 2 |
| 10 | ·9848 | 9845 | 9842 | 9839 | 9836 | 9833 | 9829 | 9826 | 9823 | 9820 | 1 | 1 | 2 | 2 | 3 |
| 11 | ·9816 | 9813 | 9810 | 9806 | 9803 | 9799 | 9796 | 9792 | 9789 | 9785 | 1 | 1 | 2 | 2 | 3 |
| 12 | ·9781 | 9778 | 9774 | 9770 | 9767 | 9763 | 9759 | 9755 | 9751 | 9748 | 1 | 1 | 2 | 3 | 3 |
| 13 | ·9744 | 9740 | 9736 | 9732 | 9728 | 9724 | 9720 | 9715 | 9711 | 9707 | 1 | 1 | 2 | 3 | 3 |
| 14 | ·9703 | 9699 | 9694 | 9690 | 9686 | 9681 | 9677 | 9673 | 9668 | 9664 | 1 | 1 | 2 | 3 | 4 |
| 15 | ·9659 | 9655 | 9650 | 9646 | 9641 | 9636 | 9632 | 9627 | 9622 | 9617 | 1 | 2 | 2 | 3 | 4 |
| 16 | ·9613 | 9608 | 9603 | 9598 | 9593 | 9588 | 9583 | 9578 | 9573 | 9568 | 1 | 2 | 2 | 3 | 4 |
| 17 | ·9563 | 9558 | 9553 | 9548 | 9542 | 9537 | 9532 | 9527 | 9521 | 9516 | 1 | 2 | 3 | 4 | 4 |
| 18 | ·9511 | 9505 | 9500 | 9494 | 9489 | 9483 | 9478 | 9472 | 9466 | 9461 | 1 | 2 | 3 | 4 | 5 |
| 19 | ·9455 | 9449 | 9444 | 9438 | 9432 | 9426 | 9421 | 9415 | 9409 | 9403 | 1 | 2 | 3 | 4 | 5 |
| 20 | ·9397 | 9391 | 9385 | 9379 | 9373 | 9367 | 9361 | 9354 | 9348 | 9342 | 1 | 2 | 3 | 4 | 5 |
| 21 | ·9336 | 9330 | 9323 | 9317 | 9311 | 9304 | 9298 | 9291 | 9285 | 9278 | 1 | 2 | 3 | 4 | 5 |
| 22 | ·9272 | 9265 | 9259 | 9252 | 9245 | 9239 | 9232 | 9225 | 9219 | 9212 | 1 | 2 | 3 | 4 | 6 |
| 23 | ·9205 | 9198 | 9191 | 9184 | 9178 | 9171 | 9164 | 9157 | 9150 | 9143 | 1 | 2 | 3 | 5 | 6 |
| 24 | ·9135 | 9128 | 9121 | 9114 | 9107 | 9100 | 9092 | 9085 | 9078 | 9070 | 1 | 2 | 4 | 5 | 6 |
| 25 | ·9063 | 9056 | 9048 | 9041 | 9033 | 9026 | 9018 | 9011 | 9003 | 8996 | 1 | 3 | 4 | 5 | 6 |
| 26 | ·8988 | 8980 | 8973 | 8965 | 8957 | 8949 | 8942 | 8934 | 8926 | 8918 | 1 | 3 | 4 | 5 | 6 |
| 27 | ·8910 | 8902 | 8894 | 8886 | 8878 | 8870 | 8862 | 8854 | 8846 | 8838 | 1 | 3 | 4 | 5 | 7 |
| 28 | ·8829 | 8821 | 8813 | 8805 | 8796 | 8788 | 8780 | 8771 | 8763 | 8755 | 1 | 3 | 4 | 6 | 7 |
| 29 | ·8746 | 8738 | 8729 | 8721 | 8712 | 8704 | 8695 | 8686 | 8678 | 8669 | 1 | 3 | 4 | 6 | 7 |
| 30 | ·8660 | 8652 | 8643 | 8634 | 8625 | 8616 | 8607 | 8599 | 8590 | 8581 | 1 | 3 | 4 | 6 | 7 |
| 31 | ·8572 | 8563 | 8554 | 8545 | 8536 | 8526 | 8517 | 8508 | 8499 | 8490 | 2 | 3 | 5 | 6 | 8 |
| 32 | ·8480 | 8471 | 8462 | 8453 | 8443 | 8434 | 8425 | 8415 | 8406 | 8396 | 2 | 3 | 5 | 6 | 8 |
| 33 | ·8387 | 8377 | 8368 | 8358 | 8348 | 8339 | 8329 | 8320 | 8310 | 8300 | 2 | 3 | 5 | 6 | 8 |
| 34 | ·8290 | 8281 | 8271 | 8261 | 8251 | 8241 | 8231 | 8221 | 8211 | 8202 | 2 | 3 | 5 | 7 | 8 |
| 35 | ·8192 | 8181 | 8171 | 8161 | 8151 | 8141 | 8131 | 8121 | 8111 | 8100 | 2 | 3 | 5 | 7 | 8 |
| 36 | ·8090 | 8080 | 8070 | 8059 | 8049 | 8039 | 8028 | 8018 | 8007 | 7997 | 2 | 3 | 5 | 7 | 9 |
| 37 | ·7986 | 7976 | 7965 | 7955 | 7944 | 7934 | 7923 | 7912 | 7902 | 7891 | 2 | 4 | 5 | 7 | 9 |
| 38 | ·7880 | 7869 | 7859 | 7848 | 7837 | 7826 | 7815 | 7804 | 7793 | 7782 | 2 | 4 | 5 | 7 | 9 |
| 39 | ·7771 | 7760 | 7749 | 7738 | 7727 | 7716 | 7705 | 7694 | 7683 | 7672 | 2 | 4 | 6 | 7 | 9 |
| 40 | ·7660 | 7649 | 7638 | 7627 | 7615 | 7604 | 7593 | 7581 | 7570 | 7559 | 2 | 4 | 6 | 8 | 9 |
| 41 | ·7547 | 7536 | 7524 | 7513 | 7501 | 7490 | 7478 | 7466 | 7455 | 7443 | 2 | 4 | 6 | 8 | 10 |
| 42 | ·7431 | 7420 | 7408 | 7396 | 7385 | 7373 | 7361 | 7349 | 7337 | 7325 | 2 | 4 | 6 | 8 | 10 |
| 43 | ·7314 | 7302 | 7290 | 7278 | 7266 | 7254 | 7242 | 7230 | 7218 | 7206 | 2 | 4 | 6 | 8 | 10 |
| 44 | ·7193 | 7181 | 7169 | 7157 | 7145 | 7133 | 7120 | 7108 | 7096 | 7083 | 2 | 4 | 6 | 8 | 10 |

# NATURAL COSINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Subtract Differences | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1′ | 2′ | 3′ | 4′ | 5′ |
| 45 | ·7071 | 7059 | 7046 | 7034 | 7022 | 7009 | 6997 | 6984 | 6972 | 6959 | 2 | 4 | 6 | 8 | 10 |
| 46 | ·6947 | 6934 | 6921 | 6909 | 6896 | 6884 | 6871 | 6858 | 6845 | 6833 | 2 | 4 | 6 | 8 | 11 |
| 47 | ·6820 | 6807 | 6794 | 6782 | 6769 | 6756 | 6743 | 6730 | 6717 | 6704 | 2 | 4 | 6 | 9 | 11 |
| 48 | ·6691 | 6678 | 6665 | 6652 | 6639 | 6626 | 6613 | 6600 | 6587 | 6574 | 2 | 4 | 7 | 9 | 11 |
| 49 | ·6561 | 6547 | 6534 | 6521 | 6508 | 6494 | 6481 | 6468 | 6455 | 6441 | 2 | 4 | 7 | 9 | 11 |
| 50 | ·6428 | 6414 | 6401 | 6388 | 6374 | 6361 | 6347 | 6334 | 6320 | 6307 | 2 | 4 | 7 | 9 | 11 |
| 51 | ·6293 | 6280 | 6266 | 6252 | 6239 | 6225 | 6211 | 6198 | 6184 | 6170 | 2 | 5 | 7 | 9 | 11 |
| 52 | ·6157 | 6143 | 6129 | 6115 | 6101 | 6088 | 6074 | 6060 | 6046 | 6032 | 2 | 5 | 7 | 9 | 12 |
| 53 | ·6018 | 6004 | 5990 | 5976 | 5962 | 5948 | 5934 | 5920 | 5906 | 5892 | 2 | 5 | 7 | 9 | 12 |
| 54 | ·5878 | 5864 | 5850 | 5835 | 5821 | 5807 | 5793 | 5779 | 5764 | 5750 | 2 | 5 | 7 | 9 | 12 |
| 55 | ·5736 | 5721 | 5707 | 5693 | 5678 | 5664 | 5650 | 5635 | 5621 | 5606 | 2 | 5 | 7 | 10 | 12 |
| 56 | ·5592 | 5577 | 5563 | 5548 | 5534 | 5519 | 5505 | 5490 | 5476 | 5461 | 2 | 5 | 7 | 10 | 12 |
| 57 | ·5446 | 5432 | 5417 | 5402 | 5388 | 5373 | 5358 | 5344 | 5329 | 5314 | 2 | 5 | 7 | 10 | 12 |
| 58 | ·5299 | 5284 | 5270 | 5255 | 5240 | 5225 | 5210 | 5195 | 5180 | 5165 | 2 | 5 | 7 | 10 | 12 |
| 59 | ·5150 | 5135 | 5120 | 5105 | 5090 | 5075 | 5060 | 5045 | 5030 | 5015 | 3 | 5 | 8 | 10 | 13 |
| 60 | ·5000 | 4985 | 4970 | 4955 | 4939 | 4924 | 4909 | 4894 | 4879 | 4863 | 3 | 5 | 8 | 10 | 13 |
| 61 | ·4848 | 4833 | 4818 | 4802 | 4787 | 4772 | 4756 | 4741 | 4726 | 4710 | 3 | 5 | 8 | 10 | 13 |
| 62 | ·4695 | 4679 | 4664 | 4648 | 4633 | 4617 | 4602 | 4586 | 4571 | 4555 | 3 | 5 | 8 | 10 | 13 |
| 63 | ·4540 | 4524 | 4509 | 4493 | 4478 | 4462 | 4446 | 4431 | 4415 | 4399 | 3 | 5 | 8 | 10 | 13 |
| 64 | ·4384 | 4368 | 4352 | 4337 | 4321 | 4305 | 4289 | 4274 | 4258 | 4242 | 3 | 5 | 8 | 11 | 13 |
| 65 | ·4226 | 4210 | 4195 | 4179 | 4163 | 4147 | 4131 | 4115 | 4099 | 4083 | 3 | 5 | 8 | 11 | 13 |
| 66 | ·4067 | 4051 | 4035 | 4019 | 4003 | 3987 | 3971 | 3955 | 3939 | 3923 | 3 | 5 | 8 | 11 | 13 |
| 67 | ·3907 | 3891 | 3875 | 3859 | 3843 | 3827 | 3811 | 3795 | 3778 | 3762 | 3 | 5 | 8 | 11 | 13 |
| 68 | ·3746 | 3730 | 3714 | 3697 | 3681 | 3665 | 3649 | 3633 | 3616 | 3600 | 3 | 5 | 8 | 11 | 14 |
| 69 | ·3584 | 3567 | 3551 | 3535 | 3518 | 3502 | 3486 | 3469 | 3453 | 3437 | 3 | 5 | 8 | 11 | 14 |
| 70 | ·3420 | 3404 | 3387 | 3371 | 3355 | 3338 | 3322 | 3305 | 3289 | 3272 | 3 | 5 | 8 | 11 | 14 |
| 71 | ·3256 | 3239 | 3223 | 3206 | 3190 | 3173 | 3156 | 3140 | 3123 | 3107 | 3 | 6 | 8 | 11 | 14 |
| 72 | ·3090 | 3074 | 3057 | 3040 | 3024 | 3007 | 2990 | 2974 | 2957 | 2940 | 3 | 6 | 8 | 11 | 14 |
| 73 | ·2924 | 2907 | 2890 | 2874 | 2857 | 2840 | 2823 | 2807 | 2790 | 2773 | 3 | 6 | 8 | 11 | 14 |
| 74 | ·2756 | 2740 | 2723 | 2706 | 2689 | 2672 | 2656 | 2639 | 2622 | 2605 | 3 | 6 | 8 | 11 | 14 |
| 75 | ·2588 | 2571 | 2554 | 2538 | 2521 | 2504 | 2487 | 2470 | 2453 | 2436 | 3 | 6 | 8 | 11 | 14 |
| 76 | ·2419 | 2402 | 2385 | 2368 | 2351 | 2334 | 2317 | 2300 | 2284 | 2267 | 3 | 6 | 8 | 11 | 14 |
| 77 | ·2250 | 2233 | 2215 | 2198 | 2181 | 2164 | 2147 | 2130 | 2113 | 2096 | 3 | 6 | 9 | 11 | 14 |
| 78 | ·2079 | 2062 | 2045 | 2028 | 2011 | 1994 | 1977 | 1959 | 1942 | 1925 | 3 | 6 | 9 | 11 | 14 |
| 79 | ·1908 | 1891 | 1874 | 1857 | 1840 | 1822 | 1805 | 1788 | 1771 | 1754 | 3 | 6 | 9 | 11 | 14 |
| 80 | ·1736 | 1719 | 1702 | 1685 | 1668 | 1650 | 1633 | 1616 | 1599 | 1582 | 3 | 6 | 9 | 11 | 14 |
| 81 | ·1564 | 1547 | 1530 | 1513 | 1495 | 1478 | 1461 | 1444 | 1426 | 1409 | 3 | 6 | 9 | 12 | 14 |
| 82 | ·1392 | 1374 | 1357 | 1340 | 1323 | 1305 | 1288 | 1271 | 1253 | 1236 | 3 | 6 | 9 | 12 | 14 |
| 83 | ·1219 | 1201 | 1184 | 1167 | 1149 | 1132 | 1115 | 1097 | 1080 | 1063 | 3 | 6 | 9 | 12 | 14 |
| 84 | ·1045 | 1028 | 1011 | 0993 | 0976 | 0958 | 0941 | 0924 | 0906 | 0889 | 3 | 6 | 9 | 12 | 14 |
| 85 | ·0872 | 0854 | 0837 | 0819 | 0802 | 0785 | 0767 | 0750 | 0732 | 0715 | 3 | 6 | 9 | 12 | 14 |
| 86 | ·0698 | 0680 | 0663 | 0645 | 0628 | 0610 | 0593 | 0576 | 0558 | 0541 | 3 | 6 | 9 | 12 | 15 |
| 87 | ·0523 | 0506 | 0488 | 0471 | 0454 | 0436 | 0419 | 0401 | 0384 | 0366 | 3 | 6 | 9 | 12 | 15 |
| 88 | ·0349 | 0332 | 0314 | 0297 | 0279 | 0262 | 0244 | 0227 | 0209 | 0192 | 3 | 6 | 9 | 12 | 15 |
| 89 | ·0175 | 0157 | 0140 | 0122 | 0105 | 0087 | 0070 | 0052 | 0035 | 0017 | 3 | 6 | 9 | 12 | 15 |

# NATURAL TANGENTS

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ·0000 | 0017 | 0035 | 0052 | 0070 | 0087 | 0105 | 0122 | 0140 | 0157 | 3 | 6 | 9 | 12 | 15 |
| 1 | ·0175 | 0192 | 0209 | 0227 | 0244 | 0262 | 0279 | 0297 | 0314 | 0332 | 3 | 6 | 9 | 12 | 15 |
| 2 | ·0349 | 0367 | 0384 | 0402 | 0419 | 0437 | 0454 | 0472 | 0489 | 0507 | 3 | 6 | 9 | 12 | 15 |
| 3 | ·0524 | 0542 | 0559 | 0577 | 0594 | 0612 | 0629 | 0647 | 0664 | 0682 | 3 | 6 | 9 | 12 | 15 |
| 4 | ·0699 | 0717 | 0734 | 0752 | 0769 | 0787 | 0805 | 0822 | 0840 | 0857 | 3 | 6 | 9 | 12 | 15 |
| 5 | ·0875 | 0892 | 0910 | 0928 | 0945 | 0963 | 0981 | 0998 | 1016 | 1033 | 3 | 6 | 9 | 12 | 15 |
| 6 | ·1051 | 1069 | 1086 | 1104 | 1122 | 1139 | 1157 | 1175 | 1192 | 1210 | 3 | 6 | 9 | 12 | 15 |
| 7 | ·1228 | 1246 | 1263 | 1281 | 1299 | 1317 | 1334 | 1352 | 1370 | 1388 | 3 | 6 | 9 | 12 | 15 |
| 8 | ·1405 | 1423 | 1441 | 1459 | 1477 | 1495 | 1512 | 1530 | 1548 | 1566 | 3 | 6 | 9 | 12 | 15 |
| 9 | ·1584 | 1602 | 1620 | 1638 | 1655 | 1673 | 1691 | 1709 | 1727 | 1745 | 3 | 6 | 9 | 12 | 15 |
| 10 | ·1763 | 1781 | 1799 | 1817 | 1835 | 1853 | 1871 | 1890 | 1908 | 1926 | 3 | 6 | 9 | 12 | 15 |
| 11 | ·1944 | 1962 | 1980 | 1998 | 2016 | 2035 | 2053 | 2071 | 2089 | 2107 | 3 | 6 | 9 | 12 | 15 |
| 12 | ·2126 | 2144 | 2162 | 2180 | 2199 | 2217 | 2235 | 2254 | 2272 | 2290 | 3 | 6 | 9 | 12 | 15 |
| 13 | ·2309 | 2327 | 2345 | 2364 | 2382 | 2401 | 2419 | 2438 | 2456 | 2475 | 3 | 6 | 9 | 12 | 15 |
| 14 | ·2493 | 2512 | 2530 | 2549 | 2568 | 2586 | 2605 | 2623 | 2642 | 2661 | 3 | 6 | 9 | 12 | 16 |
| 15 | ·2679 | 2698 | 2717 | 2736 | 2754 | 2773 | 2792 | 2811 | 2830 | 2849 | 3 | 6 | 9 | 13 | 16 |
| 16 | ·2867 | 2886 | 2905 | 2924 | 2943 | 2962 | 2981 | 3000 | 3019 | 3038 | 3 | 6 | 9 | 13 | 16 |
| 17 | ·3057 | 3076 | 3096 | 3115 | 3134 | 3153 | 3172 | 3191 | 3211 | 3230 | 3 | 6 | 10 | 13 | 16 |
| 18 | ·3249 | 3269 | 3288 | 3307 | 3327 | 3346 | 3365 | 3385 | 3404 | 3424 | 3 | 6 | 10 | 13 | 16 |
| 19 | ·3443 | 3463 | 3482 | 3502 | 3522 | 3541 | 3561 | 3581 | 3600 | 3620 | 3 | 7 | 10 | 13 | 16 |
| 20 | ·3640 | 3659 | 3679 | 3699 | 3719 | 3739 | 3759 | 3779 | 3799 | 3819 | 3 | 7 | 10 | 13 | 17 |
| 21 | ·3839 | 3859 | 3879 | 3899 | 3919 | 3939 | 3959 | 3979 | 4000 | 4020 | 3 | 7 | 10 | 13 | 17 |
| 22 | ·4040 | 4061 | 4081 | 4101 | 4122 | 4142 | 4163 | 4183 | 4204 | 4224 | 3 | 7 | 10 | 14 | 17 |
| 23 | ·4245 | 4265 | 4286 | 4307 | 4327 | 4348 | 4369 | 4390 | 4411 | 4431 | 3 | 7 | 10 | 14 | 17 |
| 24 | ·4452 | 4473 | 4494 | 4515 | 4536 | 4557 | 4578 | 4599 | 4621 | 4642 | 4 | 7 | 11 | 14 | 18 |
| 25 | ·4663 | 4684 | 4706 | 4727 | 4748 | 4770 | 4791 | 4813 | 4834 | 4856 | 4 | 7 | 11 | 14 | 18 |
| 26 | ·4877 | 4899 | 4921 | 4942 | 4964 | 4986 | 5008 | 5029 | 5051 | 5073 | 4 | 7 | 11 | 15 | 18 |
| 27 | ·5095 | 5117 | 5139 | 5161 | 5184 | 5206 | 5228 | 5250 | 5272 | 5295 | 4 | 7 | 11 | 15 | 18 |
| 28 | ·5317 | 5340 | 5362 | 5384 | 5407 | 5430 | 5452 | 5475 | 5498 | 5520 | 4 | 8 | 11 | 15 | 19 |
| 29 | ·5543 | 5566 | 5589 | 5612 | 5635 | 5658 | 5681 | 5704 | 5727 | 5750 | 4 | 8 | 12 | 15 | 19 |
| 30 | ·5774 | 5797 | 5820 | 5844 | 5867 | 5890 | 5914 | 5938 | 5961 | 5985 | 4 | 8 | 12 | 16 | 20 |
| 31 | ·6009 | 6032 | 6056 | 6080 | 6104 | 6128 | 6152 | 6176 | 6200 | 6224 | 4 | 8 | 12 | 16 | 20 |
| 32 | ·6249 | 6273 | 6297 | 6322 | 6346 | 6371 | 6395 | 6420 | 6445 | 6469 | 4 | 8 | 12 | 16 | 20 |
| 33 | ·6494 | 6519 | 6544 | 6569 | 6594 | 6619 | 6644 | 6669 | 6694 | 6720 | 4 | 8 | 13 | 17 | 21 |
| 34 | ·6745 | 6771 | 6796 | 6822 | 6847 | 6873 | 6899 | 6924 | 6950 | 6976 | 4 | 9 | 13 | 17 | 21 |
| 35 | ·7002 | 7028 | 7054 | 7080 | 7107 | 7133 | 7159 | 7186 | 7212 | 7239 | 4 | 9 | 13 | 18 | 22 |
| 36 | ·7265 | 7292 | 7319 | 7346 | 7373 | 7400 | 7427 | 7454 | 7481 | 7508 | 5 | 9 | 14 | 18 | 23 |
| 37 | ·7536 | 7563 | 7590 | 7618 | 7646 | 7673 | 7701 | 7729 | 7757 | 7785 | 5 | 9 | 14 | 18 | 23 |
| 38 | ·7813 | 7841 | 7869 | 7898 | 7926 | 7954 | 7983 | 8012 | 8040 | 8069 | 5 | 9 | 14 | 19 | 24 |
| 39 | ·8098 | 8127 | 8156 | 8185 | 8214 | 8243 | 8273 | 8302 | 8332 | 8361 | 5 | 10 | 15 | 20 | 24 |
| 40 | ·8391 | 8421 | 8451 | 8481 | 8511 | 8541 | 8571 | 8601 | 8632 | 8662 | 5 | 10 | 15 | 20 | 25 |
| 41 | ·8693 | 8724 | 8754 | 8785 | 8816 | 8847 | 8878 | 8910 | 8941 | 8972 | 5 | 10 | 16 | 21 | 26 |
| 42 | ·9004 | 9036 | 9067 | 9099 | 9131 | 9163 | 9195 | 9228 | 9260 | 9293 | 5 | 11 | 16 | 21 | 27 |
| 43 | ·9325 | 9358 | 9391 | 9424 | 9457 | 9490 | 9523 | 9556 | 9590 | 9623 | 6 | 11 | 17 | 22 | 28 |
| 44 | ·9657 | 9691 | 9725 | 9759 | 9793 | 9827 | 9861 | 9896 | 9930 | 9965 | 6 | 11 | 17 | 23 | 29 |

# NATURAL TANGENTS

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | 1·0000 | 0035 | 0070 | 0105 | 0141 | 0176 | 0212 | 0247 | 0283 | 0319 | 6 | 12 | 18 | 24 | 30 |
| 46 | 1·0355 | 0392 | 0428 | 0464 | 0501 | 0538 | 0575 | 0612 | 0649 | 0686 | 6 | 12 | 18 | 25 | 31 |
| 47 | 1·0724 | 0761 | 0799 | 0837 | 0875 | 0913 | 0951 | 0990 | 1028 | 1067 | 6 | 13 | 19 | 25 | 32 |
| 48 | 1·1106 | 1145 | 1184 | 1224 | 1263 | 1303 | 1343 | 1383 | 1423 | 1463 | 7 | 13 | 20 | 26 | 33 |
| 49 | 1·1504 | 1544 | 1585 | 1626 | 1667 | 1708 | 1750 | 1792 | 1833 | 1875 | 7 | 14 | 21 | 28 | 34 |
| 50 | 1·1918 | 1960 | 2002 | 2045 | 2088 | 2131 | 2174 | 2218 | 2261 | 2305 | 7 | 14 | 22 | 29 | 36 |
| 51 | 1·2349 | 2393 | 2437 | 2482 | 2527 | 2572 | 2617 | 2662 | 2708 | 2753 | 8 | 15 | 23 | 30 | 38 |
| 52 | 1·2799 | 2846 | 2892 | 2938 | 2985 | 3032 | 3079 | 3127 | 3175 | 3222 | 8 | 16 | 24 | 31 | 39 |
| 53 | 1·3270 | 3319 | 3367 | 3416 | 3465 | 3514 | 3564 | 3613 | 3663 | 3713 | 8 | 16 | 25 | 33 | 41 |
| 54 | 1·3764 | 3814 | 3865 | 3916 | 3968 | 4019 | 4071 | 4124 | 4176 | 4229 | 9 | 17 | 26 | 34 | 43 |
| 55 | 1·4281 | 4335 | 4388 | 4442 | 4496 | 4550 | 4605 | 4659 | 4715 | 4770 | 9 | 18 | 27 | 36 | 45 |
| 56 | 1·4826 | 4882 | 4938 | 4994 | 5051 | 5108 | 5166 | 5224 | 5282 | 5340 | 10 | 19 | 29 | 38 | 48 |
| 57 | 1·5399 | 5458 | 5517 | 5577 | 5637 | 5697 | 5757 | 5818 | 5880 | 5941 | 10 | 20 | 30 | 40 | 50 |
| 58 | 1·6003 | 6066 | 6128 | 6191 | 6255 | 6319 | 6383 | 6447 | 6512 | 6577 | 11 | 21 | 32 | 43 | 53 |
| 59 | 1·6643 | 6709 | 6775 | 6842 | 6909 | 6977 | 7045 | 7113 | 7182 | 7251 | 11 | 23 | 34 | 45 | 56 |
| 60 | 1·7321 | 7391 | 7461 | 7532 | 7603 | 7675 | 7747 | 7820 | 7893 | 7966 | 12 | 24 | 36 | 48 | 60 |
| 61 | 1·8040 | 8115 | 8190 | 8265 | 8341 | 8418 | 8495 | 8572 | 8650 | 8728 | 13 | 26 | 38 | 51 | 64 |
| 62 | 1·8807 | 8887 | 8967 | 9047 | 9128 | 9210 | 9292 | 9375 | 9458 | 9542 | 14 | 27 | 41 | 55 | 68 |
| 63 | 1·9626 | 9711 | 9797 | 9883 | 9970 | $\bar{0}057$ | $\bar{0}145$ | $\bar{0}233$ | $\bar{0}323$ | $\bar{0}413$ | 15 | 29 | 44 | 58 | 73 |
| 64 | 2·0503 | 0594 | 0686 | 0778 | 0872 | 0965 | 1060 | 1155 | 1251 | 1348 | 16 | 31 | 47 | 63 | 78 |
| 65 | 2·1445 | 1543 | 1642 | 1742 | 1842 | 1943 | 2045 | 2148 | 2251 | 2355 | 17 | 34 | 51 | 68 | 85 |
| 66 | 2·2460 | 2566 | 2673 | 2781 | 2889 | 2998 | 3109 | 3220 | 3332 | 3445 | 18 | 37 | 55 | 73 | 91 |
| 67 | 2·3559 | 3673 | 3789 | 3906 | 4023 | 4142 | 4262 | 4383 | 4504 | 4627 | 20 | 40 | 60 | 79 | 99 |
| 68 | 2·4751 | 4876 | 5002 | 5129 | 5257 | 5386 | 5517 | 5649 | 5782 | 5916 | 22 | 43 | 65 | 87 | 108 |
| 69 | 2·6051 | 6187 | 6325 | 6464 | 6605 | 6746 | 6889 | 7034 | 7179 | 7326 | 24 | 47 | 71 | 95 | 119 |
| 70 | 2·7475 | 7625 | 7776 | 7929 | 8083 | 8239 | 8397 | 8556 | 8716 | 8878 | 26 | 52 | 78 | 104 | 131 |
| 71 | 2·9042 | 9208 | 9375 | 9544 | 9714 | 9887 | $\bar{0}061$ | $\bar{0}237$ | $\bar{0}415$ | $\bar{0}595$ | 29 | 58 | 87 | 116 | 145 |
| 72 | 3·0777 | 0961 | 1146 | 1334 | 1524 | 1716 | 1910 | 2106 | 2305 | 2506 | 32 | 64 | 96 | 129 | 161 |
| 73 | 3·2709 | 2914 | 3122 | 3332 | 3544 | 3759 | 3977 | 4197 | 4420 | 4646 | 36 | 72 | 108 | 144 | 180 |
| 74 | 3·4874 | 5105 | 5339 | 5576 | 5816 | 6059 | 6305 | 6554 | 6806 | 7062 | 41 | 81 | 122 | 163 | 204 |
| 75 | 3·7321 | 7583 | 7848 | 8118 | 8391 | 8667 | 8947 | 9232 | 9520 | 9812 | 46 | 93 | 139 | 186 | 232 |
| 76 | 4·0108 | 0408 | 0713 | 1022 | 1335 | 1653 | 1976 | 2303 | 2635 | 2972 | | | | | |
| 77 | 4·3315 | 3662 | 4015 | 4373 | 4737 | 5107 | 5483 | 5864 | 6252 | 6646 | | | | | |
| 78 | 4·7046 | 7453 | 7867 | 8288 | 8716 | 9152 | 9594 | $\bar{0}045$ | $\bar{0}504$ | $\bar{0}970$ | | | | | |
| 79 | 5·1446 | 1929 | 2422 | 2924 | 3435 | 3955 | 4486 | 5026 | 5578 | 6140 | | | | | |
| 80 | 5·6713 | 7297 | 7894 | 8502 | 9124 | 9758 | $\bar{0}405$ | $\bar{1}066$ | $\bar{1}742$ | $\bar{2}432$ | | | | | |
| 81 | 6·3138 | 3859 | 4596 | 5350 | 6122 | 6912 | 7720 | 8548 | 9395 | $\bar{0}264$ | | | | | |
| 82 | 7·1154 | 2066 | 3002 | 3962 | 4947 | 5958 | 6996 | 8062 | 9158 | $\bar{0}285$ | | | | | |
| 83 | 8·1443 | 2636 | 3863 | 5126 | 6427 | 7769 | 9152 | $\bar{0}579$ | $\bar{2}052$ | $\bar{3}572$ | | | | | |
| 84 | 9·514 | 9·677 | 9·845 | 10·02 | 10·20 | 10·39 | 10·58 | 10·78 | 10·99 | 11·20 | | | | | |
| 85 | 11·43 | 11·66 | 11·91 | 12·16 | 12·43 | 12·71 | 13·00 | 13·30 | 13·62 | 13·95 | | | | | |
| 86 | 14·30 | 14·67 | 15·06 | 15·46 | 15·89 | 16·35 | 16·83 | 17·34 | 17·89 | 18·46 | | | | | |
| 87 | 19·08 | 19·74 | 20·45 | 21·20 | 22·02 | 22·90 | 23·86 | 24·90 | 26·03 | 27·27 | | | | | |
| 88 | 28·64 | 30·14 | 31·82 | 33·69 | 35·80 | 38·19 | 40·92 | 44·07 | 47·74 | 52·08 | | | | | |
| 89 | 57·29 | 63·66 | 71·62 | 81·85 | 95·49 | 114·6 | 143·2 | 191·0 | 286·5 | 573·0 | | | | | |

# LOG. SINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | Differences | | | | |
| **0** | $-\infty$ | $\bar{3}$·242 | $\bar{3}$·543 | $\bar{3}$·719 | $\bar{3}$·844 | $\bar{3}$·941 | $\bar{2}$·020 | $\bar{2}$·087 | $\bar{2}$·145 | $\bar{2}$·196 | | | | | |
| **1** | $\bar{2}$·2419 | 2832 | 3210 | 3558 | 3880 | 4179 | 4459 | 4723 | 4971 | 5206 | | | | | |
| **2** | $\bar{2}$·5428 | 5640 | 5842 | 6035 | 6220 | 6397 | 6567 | 6731 | 6889 | 7041 | | | | | |
| **3** | $\bar{2}$·7188 | 7330 | 7468 | 7602 | 7731 | 7857 | 7979 | 8098 | 8213 | 8326 | | | | | |
| **4** | $\bar{2}$·8436 | 8543 | 8647 | 8749 | 8849 | 8936 | 9042 | 9135 | 9226 | 9315 | | | | | |
| **5** | $\bar{2}$·9403 | 9489 | 9573 | 9655 | 9736 | 9816 | 9894 | 9970 | $\bar{0}$046 | $\bar{0}$120 | 13 | 26 | 39 | 52 | 66 |
| **6** | $\bar{1}$·0192 | 0264 | 0334 | 0403 | 0472 | 0539 | 0605 | 0670 | 0734 | 0797 | 11 | 22 | 33 | 44 | 55 |
| **7** | $\bar{1}$·0859 | 0920 | 0981 | 1040 | 1099 | 1157 | 1214 | 1271 | 1326 | 1381 | 10 | 19 | 29 | 38 | 48 |
| **8** | $\bar{1}$·1436 | 1489 | 1542 | 1594 | 1646 | 1697 | 1747 | 1797 | 1847 | 1895 | 8 | 17 | 25 | 34 | 42 |
| **9** | $\bar{1}$·1943 | 1991 | 2038 | 2085 | 2131 | 2176 | 2221 | 2266 | 2310 | 2353 | 8 | 15 | 23 | 30 | 38 |
| **10** | $\bar{1}$·2397 | 2439 | 2482 | 2524 | 2565 | 2606 | 2647 | 2687 | 2727 | 2767 | 7 | 14 | 20 | 27 | 34 |
| **11** | $\bar{1}$·2806 | 2845 | 2883 | 2921 | 2959 | 2997 | 3034 | 3070 | 3107 | 3143 | 6 | 12 | 19 | 25 | 31 |
| **12** | $\bar{1}$·3179 | 3214 | 3250 | 3284 | 3319 | 3353 | 3387 | 3421 | 3455 | 3488 | 6 | 11 | 17 | 23 | 28 |
| **13** | $\bar{1}$·3521 | 3554 | 3586 | 3618 | 3650 | 3682 | 3713 | 3745 | 3775 | 3806 | 5 | 11 | 16 | 21 | 26 |
| **14** | $\bar{1}$·3837 | 3867 | 3897 | 3927 | 3957 | 3986 | 4015 | 4044 | 4073 | 4102 | 5 | 10 | 15 | 20 | 24 |
| **15** | $\bar{1}$·4130 | 4158 | 4186 | 4214 | 4242 | 4269 | 4296 | 4323 | 4350 | 4377 | 5 | 9 | 14 | 18 | 23 |
| **16** | $\bar{1}$·4403 | 4430 | 4456 | 4482 | 4508 | 4533 | 4559 | 4584 | 4609 | 4634 | 4 | 9 | 13 | 17 | 21 |
| **17** | $\bar{1}$·4659 | 4684 | 4709 | 4733 | 4757 | 4781 | 4805 | 4829 | 4853 | 4876 | 4 | 8 | 12 | 16 | 20 |
| **18** | $\bar{1}$·4900 | 4923 | 4946 | 4969 | 4992 | 5015 | 5037 | 5060 | 5082 | 5104 | 4 | 8 | 11 | 15 | 19 |
| **19** | $\bar{1}$·5126 | 5148 | 5170 | 5192 | 5213 | 5235 | 5256 | 5278 | 5299 | 5320 | 4 | 7 | 11 | 14 | 18 |
| **20** | $\bar{1}$·5341 | 5361 | 5382 | 5402 | 5423 | 5443 | 5463 | 5484 | 5504 | 5523 | 3 | 7 | 10 | 14 | 17 |
| **21** | $\bar{1}$·5543 | 5563 | 5583 | 5602 | 5621 | 5641 | 5660 | 5679 | 5698 | 5717 | 3 | 6 | 10 | 13 | 16 |
| **22** | $\bar{1}$·5736 | 5754 | 5773 | 5792 | 5810 | 5828 | 5847 | 5865 | 5883 | 5901 | 3 | 6 | 9 | 12 | 15 |
| **23** | $\bar{1}$·5919 | 5937 | 5954 | 5972 | 5990 | 6007 | 6024 | 6042 | 6059 | 6076 | 3 | 6 | 9 | 12 | 15 |
| **24** | $\bar{1}$·6093 | 6110 | 6127 | 6144 | 6161 | 6177 | 6194 | 6210 | 6227 | 6243 | 3 | 6 | 8 | 11 | 14 |
| **25** | $\bar{1}$·6259 | 6276 | 6292 | 6308 | 6324 | 6340 | 6356 | 6371 | 6387 | 6403 | 3 | 5 | 8 | 11 | 13 |
| **26** | $\bar{1}$·6418 | 6434 | 6449 | 6465 | 6480 | 6495 | 6510 | 6526 | 6541 | 6556 | 3 | 5 | 8 | 10 | 13 |
| **27** | $\bar{1}$·6570 | 6585 | 6600 | 6615 | 6629 | 6644 | 6659 | 6673 | 6687 | 6702 | 2 | 5 | 7 | 10 | 12 |
| **28** | $\bar{1}$·6716 | 6730 | 6744 | 6759 | 6773 | 6787 | 6801 | 6814 | 6828 | 6842 | 2 | 5 | 7 | 9 | 12 |
| **29** | $\bar{1}$·6856 | 6869 | 6883 | 6896 | 6910 | 6923 | 6937 | 6950 | 6963 | 6977 | 2 | 4 | 7 | 9 | 11 |
| **30** | $\bar{1}$·6990 | 7003 | 7016 | 7029 | 7042 | 7055 | 7068 | 7080 | 7093 | 7106 | 2 | 4 | 6 | 9 | 11 |
| **31** | $\bar{1}$·7118 | 7131 | 7144 | 7156 | 7168 | 7181 | 7193 | 7205 | 7218 | 7230 | 2 | 4 | 6 | 8 | 10 |
| **32** | $\bar{1}$·7242 | 7254 | 7266 | 7278 | 7290 | 7302 | 7314 | 7326 | 7338 | 7349 | 2 | 4 | 6 | 8 | 10 |
| **33** | $\bar{1}$·7361 | 7373 | 7384 | 7396 | 7407 | 7419 | 7430 | 7442 | 7453 | 7464 | 2 | 4 | 6 | 8 | 10 |
| **34** | $\bar{1}$·7476 | 7487 | 7498 | 7509 | 7520 | 7531 | 7542 | 7553 | 7564 | 7575 | 2 | 4 | 6 | 7 | 9 |
| **35** | $\bar{1}$·7586 | 7597 | 7607 | 7618 | 7629 | 7640 | 7650 | 7661 | 7671 | 7682 | 2 | 4 | 5 | 7 | 9 |
| **36** | $\bar{1}$·7692 | 7703 | 7713 | 7723 | 7734 | 7744 | 7754 | 7764 | 7774 | 7785 | 2 | 3 | 5 | 7 | 9 |
| **37** | $\bar{1}$·7795 | 7805 | 7815 | 7825 | 7835 | 7844 | 7854 | 7864 | 7874 | 7884 | 2 | 3 | 5 | 7 | 8 |
| **38** | $\bar{1}$·7893 | 7903 | 7913 | 7922 | 7932 | 7941 | 7951 | 7960 | 7970 | 7979 | 2 | 3 | 5 | 6 | 8 |
| **39** | $\bar{1}$·7989 | 7998 | 8007 | 8017 | 8026 | 8035 | 8044 | 8053 | 8063 | 8072 | 2 | 3 | 5 | 6 | 8 |
| **40** | $\bar{1}$·8081 | 8090 | 8099 | 8108 | 8117 | 8125 | 8134 | 8143 | 8152 | 8161 | 1 | 3 | 4 | 6 | 7 |
| **41** | $\bar{1}$·8169 | 8178 | 8187 | 8195 | 8204 | 8213 | 8221 | 8230 | 8238 | 8247 | 1 | 3 | 4 | 6 | 7 |
| **42** | $\bar{1}$·8255 | 8264 | 8272 | 8280 | 8289 | 8297 | 8305 | 8313 | 8322 | 8330 | 1 | 3 | 4 | 6 | 7 |
| **43** | $\bar{1}$·8338 | 8346 | 8354 | 8362 | 8370 | 8378 | 8386 | 8394 | 8402 | 8410 | 1 | 3 | 4 | 5 | 7 |
| **44** | $\bar{1}$·8418 | 8426 | 8433 | 8441 | 8449 | 8457 | 8464 | 8472 | 8480 | 8487 | 1 | 3 | 4 | 5 | 6 |

# LOG. SINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | $\bar{1}$·8495 | 8502 | 8510 | 8517 | 8525 | 8532 | 8540 | 8547 | 8555 | 8562 | 1 | 2 | 4 | 5 | 6 |
| 46 | $\bar{1}$·8569 | 8577 | 8584 | 8591 | 8598 | 8606 | 8613 | 8620 | 8627 | 8634 | 1 | 2 | 4 | 5 | 6 |
| 47 | $\bar{1}$·8641 | 8648 | 8655 | 8662 | 8669 | 8676 | 8683 | 8690 | 8697 | 8704 | 1 | 2 | 3 | 5 | 6 |
| 48 | $\bar{1}$·8711 | 8718 | 8724 | 8731 | 8738 | 8745 | 8751 | 8758 | 8765 | 8771 | 1 | 2 | 3 | 4 | 6 |
| 49 | $\bar{1}$·8778 | 8784 | 8791 | 8797 | 8804 | 8810 | 8817 | 8823 | 8830 | 8836 | 1 | 2 | 3 | 4 | 5 |
| 50 | $\bar{1}$·8843 | 8849 | 8855 | 8862 | 8868 | 8874 | 8880 | 8887 | 8893 | 8899 | 1 | 2 | 3 | 4 | 5 |
| 51 | $\bar{1}$·8905 | 8911 | 8917 | 8923 | 8929 | 8935 | 8941 | 8947 | 8953 | 8959 | 1 | 2 | 3 | 4 | 5 |
| 52 | $\bar{1}$·8965 | 8971 | 8977 | 8983 | 8989 | 8995 | 9000 | 9006 | 9012 | 9018 | 1 | 2 | 3 | 4 | 5 |
| 53 | $\bar{1}$·9023 | 9029 | 9035 | 9041 | 9046 | 9052 | 9057 | 9063 | 9069 | 9074 | 1 | 2 | 3 | 4 | 5 |
| 54 | $\bar{1}$·9080 | 9085 | 9091 | 9096 | 9101 | 9107 | 9112 | 9118 | 9123 | 9128 | 1 | 2 | 3 | 4 | 5 |
| 55 | $\bar{1}$·9134 | 9139 | 9144 | 9149 | 9155 | 9160 | 9165 | 9170 | 9175 | 9181 | 1 | 2 | 3 | 3 | 4 |
| 56 | $\bar{1}$·9186 | 9191 | 9196 | 9201 | 9206 | 9211 | 9216 | 9221 | 9226 | 9231 | 1 | 2 | 3 | 3 | 4 |
| 57 | $\bar{1}$·9236 | 9241 | 9246 | 9251 | 9255 | 9260 | 9265 | 9270 | 9275 | 9279 | 1 | 2 | 2 | 3 | 4 |
| 58 | $\bar{1}$·9284 | 9289 | 9294 | 9298 | 9303 | 9308 | 9312 | 9317 | 9322 | 9326 | 1 | 2 | 2 | 3 | 4 |
| 59 | $\bar{1}$·9331 | 9335 | 9340 | 9344 | 9349 | 9353 | 9358 | 9362 | 9367 | 9371 | 1 | 1 | 2 | 3 | 4 |
| 60 | $\bar{1}$·9375 | 9380 | 9384 | 9388 | 9393 | 9397 | 9401 | 9406 | 9410 | 9414 | 1 | 1 | 2 | 3 | 4 |
| 61 | $\bar{1}$·9418 | 9422 | 9427 | 9431 | 9435 | 9439 | 9443 | 9447 | 9451 | 9455 | 1 | 1 | 2 | 3 | 3 |
| 62 | $\bar{1}$·9459 | 9463 | 9467 | 9471 | 9475 | 9479 | 9483 | 9487 | 9491 | 9495 | 1 | 1 | 2 | 3 | 3 |
| 63 | $\bar{1}$·9499 | 9503 | 9506 | 9510 | 9514 | 9518 | 9522 | 9525 | 9529 | 9533 | 1 | 1 | 2 | 3 | 3 |
| 64 | $\bar{1}$·9537 | 9540 | 9544 | 9548 | 9551 | 9555 | 9558 | 9562 | 9566 | 9569 | 1 | 1 | 2 | 2 | 3 |
| 65 | $\bar{1}$·9573 | 9576 | 9580 | 9583 | 9587 | 9590 | 9594 | 9597 | 9601 | 9604 | 1 | 1 | 2 | 2 | 3 |
| 66 | $\bar{1}$·9607 | 9611 | 9614 | 9617 | 9621 | 9624 | 9627 | 9631 | 9634 | 9637 | 1 | 1 | 2 | 2 | 3 |
| 67 | $\bar{1}$·9640 | 9643 | 9647 | 9650 | 9653 | 9656 | 9659 | 9662 | 9666 | 9669 | 1 | 1 | 2 | 2 | 3 |
| 68 | $\bar{1}$·9672 | 9675 | 9678 | 9681 | 9684 | 9687 | 9690 | 9693 | 9696 | 9699 | 0 | 1 | 1 | 2 | 2 |
| 69 | $\bar{1}$·9702 | 9704 | 9707 | 9710 | 9713 | 9716 | 9719 | 9722 | 9724 | 9727 | 0 | 1 | 1 | 2 | 2 |
| 70 | $\bar{1}$·9730 | 9733 | 9735 | 9738 | 9741 | 9743 | 9746 | 9749 | 9751 | 9754 | 0 | 1 | 1 | 2 | 2 |
| 71 | $\bar{1}$·9757 | 9759 | 9762 | 9764 | 9767 | 9770 | 9772 | 9775 | 9777 | 9780 | 0 | 1 | 1 | 2 | 2 |
| 72 | $\bar{1}$·9782 | 9785 | 9787 | 9789 | 9792 | 9794 | 9797 | 9799 | 9801 | 9804 | 0 | 1 | 1 | 2 | 2 |
| 73 | $\bar{1}$·9806 | 9808 | 9811 | 9813 | 9815 | 9817 | 9820 | 9822 | 9824 | 9826 | 0 | 1 | 1 | 1 | 2 |
| 74 | $\bar{1}$·9828 | 9831 | 9833 | 9835 | 9837 | 9839 | 9841 | 9843 | 9845 | 9847 | 0 | 1 | 1 | 1 | 2 |
| 75 | $\bar{1}$·9849 | 9851 | 9853 | 9855 | 9857 | 9859 | 9861 | 9863 | 9865 | 9867 | 0 | 1 | 1 | 1 | 2 |
| 76 | $\bar{1}$·9869 | 9871 | 9873 | 9875 | 9876 | 9878 | 9880 | 9882 | 9884 | 9885 | 0 | 1 | 1 | 1 | 2 |
| 77 | $\bar{1}$·9887 | 9889 | 9891 | 9892 | 9894 | 9896 | 9897 | 9899 | 9901 | 9902 | 0 | 1 | 1 | 1 | 1 |
| 78 | $\bar{1}$·9904 | 9906 | 9907 | 9909 | 9910 | 9912 | 9913 | 9915 | 9916 | 9918 | 0 | 1 | 1 | 1 | 1 |
| 79 | $\bar{1}$·9919 | 9921 | 9922 | 9924 | 9925 | 9927 | 9928 | 9929 | 9931 | 9932 | 0 | 0 | 1 | 1 | 1 |
| 80 | $\bar{1}$·9934 | 9935 | 9936 | 9937 | 9939 | 9940 | 9941 | 9943 | 9944 | 9945 | 0 | 0 | 1 | 1 | 1 |
| 81 | $\bar{1}$·9946 | 9947 | 9949 | 9950 | 9951 | 9952 | 9953 | 9954 | 9955 | 9956 | 0 | 0 | 1 | 1 | 1 |
| 82 | $\bar{1}$·9958 | 9959 | 9960 | 9961 | 9962 | 9963 | 9964 | 9965 | 9966 | 9967 | 0 | 0 | 0 | 1 | 1 |
| 83 | $\bar{1}$·9968 | 9968 | 9969 | 9970 | 9971 | 9972 | 9973 | 9974 | 9975 | 9975 | | | | | |
| 84 | $\bar{1}$·9976 | 9977 | 9978 | 9978 | 9979 | 9980 | 9981 | 9981 | 9982 | 9983 | | | | | |
| 85 | $\bar{1}$·9983 | 9984 | 9985 | 9985 | 9986 | 9987 | 9987 | 9988 | 9988 | 9989 | | | | | |
| 86 | $\bar{1}$·9989 | 9990 | 9990 | 9991 | 9991 | 9992 | 9992 | 9993 | 9993 | 9994 | | | | | |
| 87 | $\bar{1}$·9994 | 9994 | 9995 | 9995 | 9996 | 9996 | 9996 | 9996 | 9997 | 9997 | | | | | |
| 88 | $\bar{1}$·9997 | 9998 | 9998 | 9998 | 9998 | 9999 | 9999 | 9999 | 9999 | 9999 | | | | | |
| 89 | $\bar{1}$·9999 | 9999 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | $\bar{0}$000 | | | | | |

# LOG. COSINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Subtract Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0·0000 | 0000 | 0000 | 0000 | 0000 | 0000 | 0000 | 0000 | 0000 | $\bar{9}999$ | | | | | |
| 1 | $\bar{1}$·9999 | 9999 | 9999 | 9999 | 9999 | 9999 | 9998 | 9998 | 9998 | 9998 | | | | | |
| 2 | $\bar{1}$·9997 | 9997 | 9997 | 9996 | 9996 | 9996 | 9996 | 9995 | 9995 | 9994 | | | | | |
| 3 | $\bar{1}$·9994 | 9994 | 9993 | 9993 | 9992 | 9992 | 9991 | 9991 | 9990 | 9990 | | | | | |
| 4 | $\bar{1}$·9989 | 9989 | 9988 | 9988 | 9987 | 9987 | 9986 | 9985 | 9985 | 9984 | | | | | |
| 5 | $\bar{1}$·9983 | 9983 | 9982 | 9981 | 9981 | 9980 | 9979 | 9978 | 9978 | 9977 | | | | | |
| 6 | $\bar{1}$·9976 | 9975 | 9975 | 9974 | 9973 | 9972 | 9971 | 9970 | 9969 | 9968 | | | | | |
| 7 | $\bar{1}$·9968 | 9967 | 9966 | 9965 | 9964 | 9963 | 9962 | 9961 | 9960 | 9959 | 0 | 0 | 0 | 1 | 1 |
| 8 | $\bar{1}$·9958 | 9956 | 9955 | 9954 | 9953 | 9952 | 9951 | 9950 | 9949 | 9947 | 0 | 0 | 1 | 1 | 1 |
| 9 | $\bar{1}$·9946 | 9945 | 9944 | 9943 | 9941 | 9940 | 9939 | 9937 | 9936 | 9935 | 0 | 0 | 1 | 1 | 1 |
| 10 | $\bar{1}$·9934 | 9932 | 9931 | 9929 | 9928 | 9927 | 9925 | 9924 | 9922 | 9921 | 0 | 0 | 1 | 1 | 1 |
| 11 | $\bar{1}$·9919 | 9918 | 9916 | 9915 | 9913 | 9912 | 9910 | 9909 | 9907 | 9906 | 0 | 1 | 1 | 1 | 1 |
| 12 | $\bar{1}$·9904 | 9902 | 9901 | 9899 | 9897 | 9896 | 9894 | 9892 | 9891 | 9889 | 0 | 1 | 1 | 1 | 1 |
| 13 | $\bar{1}$·9887 | 9885 | 9884 | 9882 | 9880 | 9878 | 9876 | 9875 | 9873 | 9871 | 0 | 1 | 1 | 1 | 2 |
| 14 | $\bar{1}$·9869 | 9867 | 9865 | 9863 | 9861 | 9859 | 9857 | 9855 | 9853 | 9851 | 0 | 1 | 1 | 1 | 2 |
| 15 | $\bar{1}$·9849 | 9847 | 9845 | 9843 | 9841 | 9839 | 9837 | 9835 | 9833 | 9831 | 0 | 1 | 1 | 1 | 2 |
| 16 | $\bar{1}$·9828 | 9826 | 9824 | 9822 | 9820 | 9817 | 9815 | 9813 | 9811 | 9808 | 0 | 1 | 1 | 1 | 2 |
| 17 | $\bar{1}$·9806 | 9804 | 9801 | 9799 | 9797 | 9794 | 9792 | 9789 | 9787 | 9785 | 0 | 1 | 1 | 2 | 2 |
| 18 | $\bar{1}$·9782 | 9780 | 9777 | 9775 | 9772 | 9770 | 9767 | 9764 | 9762 | 9759 | 0 | 1 | 1 | 2 | 2 |
| 19 | $\bar{1}$·9757 | 9754 | 9751 | 9749 | 9746 | 9743 | 9741 | 9738 | 9735 | 9733 | 0 | 1 | 1 | 2 | 2 |
| 20 | $\bar{1}$·9730 | 9727 | 9724 | 9722 | 9719 | 9716 | 9713 | 9710 | 9707 | 9704 | 0 | 1 | 1 | 2 | 2 |
| 21 | $\bar{1}$·9702 | 9699 | 9696 | 9693 | 9690 | 9687 | 9684 | 9681 | 9678 | 9675 | 0 | 1 | 1 | 2 | 2 |
| 22 | $\bar{1}$·9672 | 9669 | 9666 | 9662 | 9659 | 9656 | 9653 | 9650 | 9647 | 9643 | 1 | 1 | 2 | 2 | 3 |
| 23 | $\bar{1}$·9640 | 9637 | 9634 | 9631 | 9627 | 9624 | 9621 | 9617 | 9614 | 9611 | 1 | 1 | 2 | 2 | 3 |
| 24 | $\bar{1}$·9607 | 9604 | 9601 | 9597 | 9594 | 9590 | 9587 | 9583 | 9580 | 9576 | 1 | 1 | 2 | 2 | 3 |
| 25 | $\bar{1}$·9573 | 9569 | 9566 | 9562 | 9558 | 9555 | 9551 | 9548 | 9544 | 9540 | 1 | 1 | 2 | 2 | 3 |
| 26 | $\bar{1}$·9537 | 9533 | 9529 | 9525 | 9522 | 9518 | 9514 | 9510 | 9506 | 9503 | 1 | 1 | 2 | 3 | 3 |
| 27 | $\bar{1}$·9499 | 9495 | 9491 | 9487 | 9483 | 9479 | 9475 | 9471 | 9467 | 9463 | 1 | 1 | 2 | 3 | 3 |
| 28 | $\bar{1}$·9459 | 9455 | 9451 | 9447 | 9443 | 9439 | 9435 | 9431 | 9427 | 9422 | 1 | 1 | 2 | 3 | 3 |
| 29 | $\bar{1}$·9418 | 9414 | 9410 | 9406 | 9401 | 9397 | 9393 | 9388 | 9384 | 9380 | 1 | 1 | 2 | 3 | 4 |
| 30 | $\bar{1}$·9375 | 9371 | 9367 | 9362 | 9358 | 9353 | 9349 | 9344 | 9340 | 9335 | 1 | 1 | 2 | 3 | 4 |
| 31 | $\bar{1}$·9331 | 9326 | 9322 | 9317 | 9312 | 9308 | 9303 | 9298 | 9294 | 9289 | 1 | 2 | 2 | 3 | 4 |
| 32 | $\bar{1}$·9284 | 9279 | 9275 | 9270 | 9265 | 9260 | 9255 | 9251 | 9246 | 9241 | 1 | 2 | 2 | 3 | 4 |
| 33 | $\bar{1}$·9236 | 9231 | 9226 | 9221 | 9216 | 9211 | 9206 | 9201 | 9196 | 9191 | 1 | 2 | 3 | 3 | 4 |
| 34 | $\bar{1}$·9186 | 9181 | 9175 | 9170 | 9165 | 9160 | 9155 | 9149 | 9144 | 9139 | 1 | 2 | 3 | 3 | 4 |
| 35 | $\bar{1}$·9134 | 9128 | 9123 | 9118 | 9112 | 9107 | 9101 | 9096 | 9091 | 9085 | 1 | 2 | 3 | 4 | 5 |
| 36 | $\bar{1}$·9080 | 9074 | 9069 | 9063 | 9057 | 9052 | 9046 | 9041 | 9035 | 9029 | 1 | 2 | 3 | 4 | 5 |
| 37 | $\bar{1}$·9023 | 9018 | 9012 | 9006 | 9000 | 8995 | 8989 | 8983 | 8977 | 8971 | 1 | 2 | 3 | 4 | 5 |
| 38 | $\bar{1}$·8965 | 8959 | 8953 | 8947 | 8941 | 8935 | 8929 | 8923 | 8917 | 8911 | 1 | 2 | 3 | 4 | 5 |
| 39 | $\bar{1}$·8905 | 8899 | 8893 | 8887 | 8880 | 8874 | 8868 | 8862 | 8855 | 8849 | 1 | 2 | 3 | 4 | 5 |
| 40 | $\bar{1}$·8843 | 8836 | 8830 | 8823 | 8817 | 8810 | 8804 | 8797 | 8791 | 8784 | 1 | 2 | 3 | 4 | 5 |
| 41 | $\bar{1}$·8778 | 8771 | 8765 | 8758 | 8751 | 8745 | 8738 | 8731 | 8724 | 8718 | 1 | 2 | 3 | 5 | 6 |
| 42 | $\bar{1}$·8711 | 8704 | 8697 | 8690 | 8683 | 8676 | 8669 | 8662 | 8655 | 8648 | 1 | 2 | 3 | 5 | 6 |
| 43 | $\bar{1}$·8641 | 8634 | 8627 | 8620 | 8613 | 8606 | 8598 | 8591 | 8584 | 8577 | 1 | 2 | 4 | 5 | 6 |
| 44 | $\bar{1}$·8569 | 8562 | 8555 | 8547 | 8540 | 8532 | 8525 | 8517 | 8510 | 8502 | 1 | 2 | 4 | 5 | 6 |

# LOG. COSINES

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Subtract Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 45 | $\bar{1}$·8495 | 8487 | 8480 | 8472 | 8464 | 8457 | 8449 | 8441 | 8433 | 8426 | 1 | 3 | 4 | 5 | 6 |
| 46 | $\bar{1}$·8418 | 8410 | 8402 | 8394 | 8386 | 8378 | 8370 | 8362 | 8354 | 8346 | 1 | 3 | 4 | 5 | 7 |
| 47 | $\bar{1}$·8338 | 8330 | 8322 | 8313 | 8305 | 8297 | 8289 | 8280 | 8272 | 8264 | 1 | 3 | 4 | 6 | 7 |
| 48 | $\bar{1}$·8255 | 8247 | 8238 | 8230 | 8221 | 8213 | 8204 | 8195 | 8187 | 8178 | 1 | 3 | 4 | 6 | 7 |
| 49 | $\bar{1}$·8169 | 8161 | 8152 | 8143 | 8134 | 8125 | 8117 | 8108 | 8099 | 8090 | 1 | 3 | 4 | 6 | 7 |
| 50 | $\bar{1}$·8081 | 8072 | 8063 | 8053 | 8044 | 8035 | 8026 | 8017 | 8007 | 7998 | 2 | 3 | 5 | 6 | 8 |
| 51 | $\bar{1}$·7989 | 7979 | 7970 | 7960 | 7951 | 7941 | 7932 | 7922 | 7913 | 7903 | 2 | 3 | 5 | 6 | 8 |
| 52 | $\bar{1}$·7893 | 7884 | 7874 | 7864 | 7854 | 7844 | 7835 | 7825 | 7815 | 7805 | 2 | 3 | 5 | 7 | 8 |
| 53 | $\bar{1}$·7795 | 7785 | 7774 | 7764 | 7754 | 7744 | 7734 | 7723 | 7713 | 7703 | 2 | 3 | 5 | 7 | 9 |
| 54 | $\bar{1}$·7692 | 7682 | 7671 | 7661 | 7650 | 7640 | 7629 | 7618 | 7607 | 7597 | 2 | 4 | 5 | 7 | 9 |
| 55 | $\bar{1}$·7586 | 7575 | 7564 | 7553 | 7542 | 7531 | 7520 | 7509 | 7498 | 7487 | 2 | 4 | 6 | 7 | 9 |
| 56 | $\bar{1}$·7476 | 7464 | 7453 | 7442 | 7430 | 7419 | 7407 | 7396 | 7384 | 7373 | 2 | 4 | 6 | 8 | 10 |
| 57 | $\bar{1}$·7361 | 7349 | 7338 | 7326 | 7314 | 7302 | 7290 | 7278 | 7266 | 7254 | 2 | 4 | 6 | 8 | 10 |
| 58 | $\bar{1}$·7242 | 7230 | 7218 | 7205 | 7193 | 7181 | 7168 | 7156 | 7144 | 7131 | 2 | 4 | 6 | 8 | 10 |
| 59 | $\bar{1}$·7118 | 7106 | 7093 | 7080 | 7068 | 7055 | 7042 | 7029 | 7016 | 7003 | 2 | 4 | 6 | 9 | 11 |
| 60 | $\bar{1}$·6990 | 6977 | 6963 | 6950 | 6937 | 6923 | 6910 | 6896 | 6883 | 6869 | 2 | 4 | 7 | 9 | 11 |
| 61 | $\bar{1}$·6856 | 6842 | 6828 | 6814 | 6801 | 6787 | 6773 | 6759 | 6744 | 6730 | 2 | 5 | 7 | 9 | 12 |
| 62 | $\bar{1}$·6716 | 6702 | 6687 | 6673 | 6659 | 6644 | 6629 | 6615 | 6600 | 6585 | 2 | 5 | 7 | 10 | 12 |
| 63 | $\bar{1}$·6570 | 6556 | 6541 | 6526 | 6510 | 6495 | 6480 | 6465 | 6449 | 6434 | 3 | 5 | 8 | 10 | 13 |
| 64 | $\bar{1}$·6418 | 6403 | 6387 | 6371 | 6356 | 6340 | 6324 | 6308 | 6292 | 6276 | 3 | 5 | 8 | 11 | 13 |
| 65 | $\bar{1}$·6259 | 6243 | 6227 | 6210 | 6194 | 6177 | 6161 | 6144 | 6127 | 6110 | 3 | 6 | 8 | 11 | 14 |
| 66 | $\bar{1}$·6093 | 6076 | 6059 | 6042 | 6024 | 6007 | 5990 | 5972 | 5954 | 5937 | 3 | 6 | 9 | 12 | 15 |
| 67 | $\bar{1}$·5919 | 5901 | 5883 | 5865 | 5847 | 5828 | 5810 | 5792 | 5773 | 5754 | 3 | 6 | 9 | 12 | 15 |
| 68 | $\bar{1}$·5736 | 5717 | 5698 | 5679 | 5660 | 5641 | 5621 | 5602 | 5583 | 5563 | 3 | 6 | 10 | 13 | 16 |
| 69 | $\bar{1}$·5543 | 5523 | 5504 | 5484 | 5463 | 5443 | 5423 | 5402 | 5382 | 5361 | 3 | 7 | 10 | 14 | 17 |
| 70 | $\bar{1}$·5341 | 5320 | 5299 | 5278 | 5256 | 5235 | 5213 | 5192 | 5170 | 5148 | 4 | 7 | 11 | 14 | 18 |
| 71 | $\bar{1}$·5126 | 5104 | 5082 | 5060 | 5037 | 5015 | 4992 | 4969 | 4946 | 4923 | 4 | 8 | 11 | 15 | 19 |
| 72 | $\bar{1}$·4900 | 4876 | 4853 | 4829 | 4805 | 4781 | 4757 | 4733 | 4709 | 4684 | 4 | 8 | 12 | 16 | 20 |
| 73 | $\bar{1}$·4659 | 4634 | 4609 | 4584 | 4559 | 4533 | 4508 | 4482 | 4456 | 4430 | 4 | 9 | 13 | 17 | 21 |
| 74 | $\bar{1}$·4403 | 4377 | 4350 | 4323 | 4296 | 4269 | 4242 | 4214 | 4186 | 4158 | 5 | 9 | 14 | 18 | 23 |
| 75 | $\bar{1}$·4130 | 4102 | 4073 | 4044 | 4015 | 3986 | 3957 | 3927 | 3897 | 3867 | 5 | 10 | 15 | 20 | 24 |
| 76 | $\bar{1}$·3837 | 3806 | 3775 | 3745 | 3713 | 3682 | 3650 | 3618 | 3586 | 3554 | 5 | 11 | 16 | 21 | 26 |
| 77 | $\bar{1}$·3521 | 3488 | 3455 | 3421 | 3387 | 3353 | 3319 | 3284 | 3250 | 3214 | 6 | 11 | 17 | 23 | 28 |
| 78 | $\bar{1}$·3179 | 3143 | 3107 | 3070 | 3034 | 2997 | 2959 | 2921 | 2883 | 2845 | 6 | 12 | 19 | 25 | 31 |
| 79 | $\bar{1}$·2806 | 2767 | 2727 | 2687 | 2647 | 2606 | 2565 | 2524 | 2482 | 2439 | 7 | 14 | 20 | 27 | 34 |
| 80 | $\bar{1}$·2397 | 2353 | 2310 | 2266 | 2221 | 2176 | 2131 | 2085 | 2038 | 1991 | 8 | 15 | 23 | 30 | 38 |
| 81 | $\bar{1}$·1943 | 1895 | 1847 | 1797 | 1747 | 1697 | 1646 | 1594 | 1542 | 1489 | 8 | 17 | 25 | 34 | 42 |
| 82 | $\bar{1}$·1436 | 1381 | 1326 | 1271 | 1214 | 1157 | 1099 | 1040 | 0981 | 0920 | 10 | 19 | 29 | 38 | 48 |
| 83 | $\bar{1}$·0859 | 0797 | 0734 | 0670 | 0605 | 0539 | 0472 | 0403 | 0334 | 0264 | 11 | 22 | 33 | 44 | 55 |
| 84 | $\bar{1}$·0192 | 0120 | 0046 | $\bar{9}$970 | $\bar{9}$894 | $\bar{9}$816 | $\bar{9}$736 | $\bar{9}$655 | $\bar{9}$573 | $\bar{9}$489 | 13 | 26 | 39 | 52 | 66 |
| 85 | $\bar{2}$·9403 | 9315 | 9226 | 9135 | 9042 | 8946 | 8849 | 8749 | 8647 | 8543 | | | | | |
| 86 | $\bar{2}$·8436 | 8326 | 8213 | 8098 | 7979 | 7857 | 7731 | 7602 | 7468 | 7330 | | | | | |
| 87 | $\bar{2}$·7188 | 7041 | 6889 | 6731 | 6567 | 6397 | 6220 | 6035 | 5842 | 5640 | | | | | |
| 88 | $\bar{2}$·5428 | 5206 | 4971 | 4723 | 4459 | 4179 | 3880 | 3558 | 3210 | 2832 | | | | | |
| 89 | $\bar{2}$·242 | $\bar{2}$·196 | $\bar{2}$·145 | $\bar{2}$·087 | $\bar{2}$·020 | $\bar{3}$·941 | $\bar{3}$·844 | $\bar{3}$·719 | $\bar{3}$·543 | $\bar{3}$·242 | | | | | |

# LOG. TANGENTS

| ° | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | —∞ | $\bar{3}$·242 | $\bar{3}$·543 | $\bar{3}$·719 | $\bar{3}$·844 | $\bar{3}$·941 | $\bar{2}$·020 | $\bar{2}$·087 | $\bar{2}$·145 | $\bar{2}$·196 | | | | | |
| 1 | $\bar{2}$·2419 | 2833 | 3211 | 3559 | 3881 | 4181 | 4461 | 4725 | 4973 | 5208 | | | | | |
| 2 | $\bar{2}$·5431 | 5643 | 5845 | 6038 | 6223 | 6401 | 6571 | 6736 | 6894 | 7046 | | | | | |
| 3 | $\bar{2}$·7194 | 7337 | 7475 | 7609 | 7739 | 7865 | 7988 | 8107 | 8223 | 8336 | | | | | |
| 4 | $\bar{2}$·8446 | 8554 | 8659 | 8762 | 8862 | 8960 | 9056 | 9150 | 9241 | 9331 | | | | | |
| 5 | $\bar{2}$·9420 | 9506 | 9591 | 9674 | 9756 | 9836 | 9915 | 9992 | $\bar{0}$068 | $\bar{0}$143 | 13 | 26 | 40 | 53 | 66 |
| 6 | $\bar{1}$·0216 | 0289 | 0360 | 0430 | 0499 | 0567 | 0633 | 0699 | 0764 | 0828 | 11 | 22 | 34 | 45 | 56 |
| 7 | $\bar{1}$·0891 | 0954 | 1015 | 1076 | 1135 | 1194 | 1252 | 1310 | 1367 | 1423 | 10 | 20 | 29 | 39 | 49 |
| 8 | $\bar{1}$·1478 | 1533 | 1587 | 1640 | 1693 | 1745 | 1797 | 1848 | 1898 | 1948 | 9 | 17 | 26 | 35 | 43 |
| 9 | $\bar{1}$·1997 | 2046 | 2094 | 2142 | 2189 | 2236 | 2282 | 2328 | 2374 | 2419 | 8 | 16 | 23 | 31 | 39 |
| 10 | $\bar{1}$·2463 | 2507 | 2551 | 2594 | 2637 | 2680 | 2722 | 2764 | 2805 | 2846 | 7 | 14 | 21 | 28 | 35 |
| 11 | $\bar{1}$·2887 | 2927 | 2967 | 3006 | 3046 | 3085 | 3123 | 3162 | 3200 | 3237 | 6 | 13 | 19 | 26 | 32 |
| 12 | $\bar{1}$·3275 | 3312 | 3349 | 3385 | 3422 | 3458 | 3493 | 3529 | 3564 | 3599 | 6 | 12 | 18 | 24 | 30 |
| 13 | $\bar{1}$·3634 | 3668 | 3702 | 3736 | 3770 | 3804 | 3837 | 3870 | 3903 | 3935 | 6 | 11 | 17 | 22 | 28 |
| 14 | $\bar{1}$·3968 | 4000 | 4032 | 4064 | 4095 | 4127 | 4158 | 4189 | 4220 | 4250 | 5 | 10 | 16 | 21 | 26 |
| 15 | $\bar{1}$·4281 | 4311 | 4341 | 4371 | 4400 | 4430 | 4459 | 4488 | 4517 | 4546 | 5 | 10 | 15 | 20 | 25 |
| 16 | $\bar{1}$·4575 | 4603 | 4632 | 4660 | 4688 | 4716 | 4744 | 4771 | 4799 | 4826 | 5 | 9 | 14 | 19 | 23 |
| 17 | $\bar{1}$·4853 | 4880 | 4907 | 4934 | 4961 | 4987 | 5014 | 5040 | 5066 | 5092 | 4 | 9 | 13 | 18 | 22 |
| 18 | $\bar{1}$·5118 | 5143 | 5169 | 5195 | 5220 | 5245 | 5270 | 5295 | 5320 | 5345 | 4 | 8 | 13 | 17 | 21 |
| 19 | $\bar{1}$·5370 | 5394 | 5419 | 5443 | 5467 | 5491 | 5516 | 5539 | 5563 | 5587 | 4 | 8 | 12 | 16 | 20 |
| 20 | $\bar{1}$·5611 | 5634 | 5658 | 5681 | 5704 | 5727 | 5750 | 5773 | 5796 | 5819 | 4 | 8 | 12 | 15 | 19 |
| 21 | $\bar{1}$·5842 | 5864 | 5887 | 5909 | 5932 | 5954 | 5976 | 5998 | 6020 | 6042 | 4 | 7 | 11 | 15 | 19 |
| 22 | $\bar{1}$·6064 | 6086 | 6108 | 6129 | 6151 | 6172 | 6194 | 6215 | 6236 | 6257 | 4 | 7 | 11 | 14 | 18 |
| 23 | $\bar{1}$·6279 | 6300 | 6321 | 6341 | 6362 | 6383 | 6404 | 6424 | 6445 | 6465 | 3 | 7 | 10 | 14 | 17 |
| 24 | $\bar{1}$·6486 | 6506 | 6527 | 6547 | 6567 | 6587 | 6607 | 6627 | 6647 | 6667 | 3 | 7 | 10 | 13 | 17 |
| 25 | $\bar{1}$·6687 | 6706 | 6726 | 6746 | 6765 | 6785 | 6804 | 6824 | 6843 | 6863 | 3 | 7 | 10 | 13 | 16 |
| 26 | $\bar{1}$·6882 | 6901 | 6920 | 6939 | 6958 | 6977 | 6996 | 7015 | 7034 | 7053 | 3 | 6 | 9 | 13 | 16 |
| 27 | $\bar{1}$·7072 | 7090 | 7109 | 7128 | 7146 | 7165 | 7183 | 7202 | 7220 | 7238 | 3 | 6 | 9 | 12 | 15 |
| 28 | $\bar{1}$·7257 | 7275 | 7293 | 7311 | 7330 | 7348 | 7366 | 7384 | 7402 | 7420 | 3 | 6 | 9 | 12 | 15 |
| 29 | $\bar{1}$·7438 | 7455 | 7473 | 7491 | 7509 | 7526 | 7544 | 7562 | 7579 | 7597 | 3 | 6 | 9 | 12 | 15 |
| 30 | $\bar{1}$·7614 | 7632 | 7649 | 7667 | 7684 | 7701 | 7719 | 7736 | 7753 | 7771 | 3 | 6 | 9 | 12 | 14 |
| 31 | $\bar{1}$·7788 | 7805 | 7822 | 7839 | 7856 | 7873 | 7890 | 7907 | 7924 | 7941 | 3 | 6 | 9 | 11 | 14 |
| 32 | $\bar{1}$·7958 | 7975 | 7992 | 8008 | 8025 | 8042 | 8059 | 8075 | 8092 | 8109 | 3 | 6 | 8 | 11 | 14 |
| 33 | $\bar{1}$·8125 | 8142 | 8158 | 8175 | 8191 | 8208 | 8224 | 8241 | 8257 | 8274 | 3 | 5 | 8 | 11 | 14 |
| 34 | $\bar{1}$·8290 | 8306 | 8323 | 8339 | 8355 | 8371 | 8388 | 8404 | 8420 | 8436 | 3 | 5 | 8 | 11 | 14 |
| 35 | $\bar{1}$·8452 | 8468 | 8484 | 8501 | 8517 | 8533 | 8549 | 8565 | 8581 | 8597 | 3 | 5 | 8 | 11 | 13 |
| 36 | $\bar{1}$·8613 | 8629 | 8644 | 8660 | 8676 | 8692 | 8708 | 8724 | 8740 | 8755 | 3 | 5 | 8 | 11 | 13 |
| 37 | $\bar{1}$·8771 | 8787 | 8803 | 8818 | 8834 | 8850 | 8865 | 8881 | 8897 | 8912 | 3 | 5 | 8 | 10 | 13 |
| 38 | $\bar{1}$·8928 | 8944 | 8959 | 8975 | 8990 | 9006 | 9022 | 9037 | 9053 | 9068 | 3 | 5 | 8 | 10 | 13 |
| 39 | $\bar{1}$·9084 | 9099 | 9115 | 9130 | 9146 | 9161 | 9176 | 9192 | 9207 | 9223 | 3 | 5 | 8 | 10 | 13 |
| 40 | $\bar{1}$·9238 | 9254 | 9269 | 9284 | 9300 | 9315 | 9330 | 9346 | 9361 | 9376 | 3 | 5 | 8 | 10 | 13 |
| 41 | $\bar{1}$·9392 | 9407 | 9422 | 9438 | 9453 | 9468 | 9483 | 9499 | 9514 | 9529 | 3 | 5 | 8 | 10 | 13 |
| 42 | $\bar{1}$·9544 | 9560 | 9575 | 9590 | 9605 | 9621 | 9636 | 9651 | 9666 | 9681 | 3 | 5 | 8 | 10 | 13 |
| 43 | $\bar{1}$·9697 | 9712 | 9727 | 9742 | 9757 | 9772 | 9788 | 9803 | 9818 | 9833 | 3 | 5 | 8 | 10 | 13 |
| 44 | $\bar{1}$·9848 | 9864 | 9879 | 9894 | 9909 | 9924 | 9939 | 9955 | 9970 | 9985 | 3 | 5 | 8 | 10 | 13 |

# LOG. TANGENTS

| | 0′ | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ | Differences 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ° | | | | | | | | | | | | | | | |
| 45 | ·0000 | 0015 | 0030 | 0045 | 0061 | 0076 | 0091 | 0106 | 0121 | 0136 | 3 | 5 | 8 | 10 | 13 |
| 46 | ·0152 | 0167 | 0182 | 0197 | 0212 | 0228 | 0243 | 0258 | 0273 | 0288 | 3 | 5 | 8 | 10 | 13 |
| 47 | ·0303 | 0319 | 0334 | 0349 | 0364 | 0379 | 0395 | 0410 | 0425 | 0440 | 3 | 5 | 8 | 10 | 13 |
| 48 | ·0456 | 0471 | 0486 | 0501 | 0517 | 0532 | 0547 | 0562 | 0578 | 0593 | 3 | 5 | 8 | 10 | 13 |
| 49 | ·0608 | 0624 | 0639 | 0654 | 0670 | 0685 | 0700 | 0716 | 0731 | 0746 | 3 | 5 | 8 | 10 | 13 |
| 50 | ·0762 | 0777 | 0793 | 0808 | 0824 | 0839 | 0854 | 0870 | 0885 | 0901 | 3 | 5 | 8 | 10 | 13 |
| 51 | ·0916 | 0932 | 0947 | 0963 | 0978 | 0994 | 1010 | 1025 | 1041 | 1056 | 3 | 5 | 8 | 10 | 13 |
| 52 | ·1072 | 1088 | 1103 | 1119 | 1135 | 1150 | 1166 | 1182 | 1197 | 1213 | 3 | 5 | 8 | 10 | 13 |
| 53 | ·1229 | 1245 | 1260 | 1276 | 1292 | 1308 | 1324 | 1340 | 1356 | 1371 | 3 | 5 | 8 | 11 | 13 |
| 54 | ·1387 | 1403 | 1419 | 1435 | 1451 | 1467 | 1483 | 1499 | 1516 | 1532 | 3 | 5 | 8 | 11 | 13 |
| 55 | ·1548 | 1564 | 1580 | 1596 | 1612 | 1629 | 1645 | 1661 | 1677 | 1694 | 3 | 5 | 8 | 11 | 14 |
| 56 | ·1710 | 1726 | 1743 | 1759 | 1776 | 1792 | 1809 | 1825 | 1842 | 1858 | 3 | 5 | 8 | 11 | 14 |
| 57 | ·1875 | 1891 | 1908 | 1925 | 1941 | 1958 | 1975 | 1992 | 2008 | 2025 | 3 | 6 | 8 | 11 | 14 |
| 58 | ·2042 | 2059 | 2076 | 2093 | 2110 | 2127 | 2144 | 2161 | 2178 | 2195 | 3 | 6 | 9 | 11 | 14 |
| 59 | ·2212 | 2229 | 2247 | 2264 | 2281 | 2299 | 2316 | 2333 | 2351 | 2368 | 3 | 6 | 9 | 12 | 14 |
| 60 | ·2386 | 2403 | 2421 | 2438 | 2456 | 2474 | 2491 | 2509 | 2527 | 2545 | 3 | 6 | 9 | 12 | 15 |
| 61 | ·2562 | 2580 | 2598 | 2616 | 2634 | 2652 | 2670 | 2689 | 2707 | 2725 | 3 | 6 | 9 | 12 | 15 |
| 62 | ·2743 | 2762 | 2780 | 2798 | 2817 | 2835 | 2854 | 2872 | 2801 | 2910 | 3 | 6 | 9 | 12 | 15 |
| 63 | ·2928 | 2947 | 2966 | 2985 | 3004 | 3023 | 3042 | 3061 | 3080 | 3099 | 3 | 6 | 9 | 13 | 16 |
| 64 | ·3118 | 3137 | 3157 | 3176 | 3196 | 3215 | 3235 | 3254 | 3274 | 3294 | 3 | 7 | 10 | 13 | 16 |
| 65 | ·3313 | 3333 | 3353 | 3373 | 3393 | 3413 | 3433 | 3453 | 3473 | 3494 | 3 | 7 | 10 | 13 | 17 |
| 66 | ·3514 | 3535 | 3555 | 3576 | 3596 | 3617 | 3638 | 3659 | 3679 | 3700 | 3 | 7 | 10 | 14 | 17 |
| 67 | ·3721 | 3743 | 3764 | 3785 | 3806 | 3828 | 3849 | 3871 | 3892 | 3914 | 4 | 7 | 11 | 14 | 18 |
| 68 | ·3936 | 3958 | 3980 | 4002 | 4024 | 4046 | 4068 | 4091 | 4113 | 4136 | 4 | 7 | 11 | 15 | 19 |
| 69 | ·4158 | 4181 | 4204 | 4227 | 4250 | 4273 | 4296 | 4319 | 4342 | 4366 | 4 | 8 | 12 | 15 | 19 |
| 70 | ·4389 | 4413 | 4437 | 4461 | 4484 | 4509 | 4533 | 4557 | 4581 | 4606 | 4 | 8 | 12 | 16 | 20 |
| 71 | ·4630 | 4655 | 4680 | 4705 | 4730 | 4755 | 4780 | 4805 | 4831 | 4857 | 4 | 8 | 13 | 17 | 21 |
| 72 | ·4882 | 4908 | 4934 | 4960 | 4986 | 5013 | 5039 | 5066 | 5093 | 5120 | 4 | 9 | 13 | 18 | 22 |
| 73 | ·5147 | 5174 | 5201 | 5229 | 5256 | 5284 | 5312 | 5340 | 5368 | 5397 | 5 | 9 | 14 | 19 | 23 |
| 74 | ·5425 | 5454 | 5483 | 5512 | 5541 | 5570 | 5600 | 5629 | 5659 | 5689 | 5 | 10 | 15 | 20 | 25 |
| 75 | ·5719 | 5750 | 5780 | 5811 | 5842 | 5873 | 5905 | 5936 | 5968 | 6000 | 5 | 10 | 16 | 21 | 26 |
| 76 | ·6032 | 6065 | 6097 | 6130 | 6163 | 6196 | 6230 | 6264 | 6298 | 6332 | 6 | 11 | 17 | 22 | 28 |
| 77 | ·6366 | 6401 | 6436 | 6471 | 6507 | 6542 | 6578 | 6615 | 6651 | 6688 | 6 | 12 | 18 | 24 | 30 |
| 78 | ·6725 | 6763 | 6800 | 6838 | 6877 | 6915 | 6954 | 6994 | 7033 | 7073 | 6 | 13 | 19 | 26 | 32 |
| 79 | ·7113 | 7154 | 7195 | 7236 | 7278 | 7320 | 7363 | 7406 | 7449 | 7493 | 7 | 14 | 21 | 28 | 35 |
| 80 | ·7537 | 7581 | 7626 | 7672 | 7718 | 7764 | 7811 | 7858 | 7906 | 7954 | 8 | 16 | 23 | 31 | 39 |
| 81 | ·8003 | 8052 | 8102 | 8152 | 8203 | 8255 | 8307 | 8360 | 8413 | 8467 | 9 | 17 | 26 | 35 | 43 |
| 82 | ·8522 | 8577 | 8633 | 8690 | 8748 | 8806 | 8865 | 8924 | 8985 | 9046 | 10 | 20 | 29 | 39 | 49 |
| 83 | ·9109 | 9172 | 9236 | 9301 | 9367 | 9433 | 9501 | 9570 | 9640 | 9711 | 11 | 22 | 34 | 45 | 56 |
| 84 | ·9784 | 9857 | 9932 | $\overline{0}008$ | $\overline{0}085$ | $\overline{0}164$ | $\overline{0}244$ | $\overline{0}326$ | $\overline{0}409$ | $\overline{0}494$ | 13 | 26 | 40 | 53 | 66 |
| 85 | 1·0580 | 0669 | 0759 | 0850 | 0944 | 1040 | 1138 | 1238 | 1341 | 1446 | | | | | |
| 86 | 1·1554 | 1664 | 1777 | 1893 | 2012 | 2135 | 2261 | 2391 | 2525 | 2663 | | | | | |
| 87 | 1·2806 | 2954 | 3106 | 3264 | 3429 | 3599 | 3777 | 3962 | 4155 | 4357 | | | | | |
| 88 | 1·4569 | 4792 | 5027 | 5275 | 5539 | 5819 | 6119 | 6441 | 6789 | 7167 | | | | | |
| 89 | 1·7581 | 8038 | 8550 | 9130 | 9800 | $\overline{0}591$ | $\overline{1}561$ | $\overline{2}810$ | $\overline{4}571$ | $\overline{7}581$ | | | | | |

# SQUARES

| | | | | | | | | | | | Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 1000 | 1020 | 1040 | 1061 | 1082 | 1103 | 1124 | 1145 | 1166 | 1188 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| 11 | 1210 | 1232 | 1254 | 1277 | 1300 | 1323 | 1346 | 1369 | 1392 | 1416 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 18 | 21 |
| 12 | 1440 | 1464 | 1488 | 1513 | 1538 | 1563 | 1588 | 1613 | 1638 | 1664 | 3 | 5 | 8 | 10 | 13 | 15 | 18 | 20 | 23 |
| 13 | 1690 | 1716 | 1742 | 1769 | 1796 | 1823 | 1850 | 1877 | 1904 | 1932 | 3 | 5 | 8 | 11 | 14 | 16 | 19 | 22 | 24 |
| 14 | 1960 | 1988 | 2016 | 2045 | 2074 | 2103 | 2132 | 2161 | 2190 | 2220 | 3 | 6 | 9 | 12 | 15 | 17 | 20 | 23 | 26 |
| 15 | 2250 | 2280 | 2310 | 2341 | 2372 | 2403 | 2434 | 2465 | 2496 | 2528 | 3 | 6 | 9 | 12 | 16 | 19 | 22 | 25 | 28 |
| 16 | 2560 | 2592 | 2624 | 2657 | 2690 | 2723 | 2756 | 2789 | 2822 | 2856 | 3 | 7 | 10 | 13 | 17 | 20 | 23 | 26 | 30 |
| 17 | 2890 | 2924 | 2958 | 2993 | 3028 | 3063 | 3098 | 3133 | 3168 | 3204 | 4 | 7 | 11 | 14 | 18 | 21 | 25 | 28 | 32 |
| 18 | 3240 | 3276 | 3312 | 3349 | 3386 | 3423 | 3460 | 3497 | 3534 | 3572 | 4 | 7 | 11 | 15 | 19 | 22 | 26 | 30 | 33 |
| 19 | 3610 | 3648 | 3686 | 3725 | 3764 | 3803 | 3842 | 3881 | 3920 | 3960 | 4 | 8 | 12 | 16 | 20 | 23 | 27 | 31 | 35 |
| 20 | 4000 | 4040 | 4080 | 4121 | 4162 | 4203 | 4244 | 4285 | 4326 | 4368 | 4 | 8 | 12 | 16 | 21 | 25 | 29 | 33 | 37 |
| 21 | [illegible] | [illegible] | 4494 | 4537 | 4580 | 4623 | 4666 | 4709 | 4752 | 4796 | 4 | 9 | 13 | 17 | 22 | 26 | 30 | 34 | 39 |
| 22 | [illegible] | [illegible] | 4928 | 4973 | 5018 | 5063 | 5108 | 5153 | 5198 | 5244 | 5 | 9 | 14 | 18 | 23 | 27 | 32 | 36 | 41 |
| 23 | 5290 | 5336 | 5382 | 5429 | 5476 | 5523 | 5570 | 5617 | 5664 | 5712 | 5 | 9 | 14 | 19 | 24 | 28 | 33 | 38 | 42 |
| 24 | 5760 | 5808 | 5856 | 5905 | 5954 | 6003 | 6052 | 6101 | 6150 | 6200 | 5 | 10 | 15 | 20 | 25 | 29 | 34 | 39 | 44 |
| 25 | 6250 | 6300 | 6350 | 6401 | 6452 | 6503 | 6554 | 6605 | 6656 | 6708 | 5 | 10 | 15 | 20 | 26 | 31 | 36 | 41 | 46 |
| 26 | 6760 | 6812 | 6864 | 6917 | 6970 | 7023 | 7076 | 7129 | 7182 | 7236 | 5 | 11 | 16 | 21 | 27 | 32 | 37 | 42 | 48 |
| 27 | 7290 | 7344 | 7398 | 7453 | 7508 | 7563 | 7618 | 7673 | 7728 | 7784 | 6 | 11 | 17 | 22 | 28 | 33 | 39 | 44 | 50 |
| 28 | 7840 | 7896 | 7952 | 8009 | 8066 | 8123 | 8180 | 8237 | 8294 | 8352 | 6 | 11 | 17 | 23 | 29 | 34 | 40 | 46 | 51 |
| 29 | 8410 | 8468 | 8526 | 8585 | 8644 | 8703 | 8762 | 8821 | 8880 | 8940 | 6 | 12 | 18 | 24 | 30 | 35 | 41 | 47 | 53 |
| 30 | 9000 | 9060 | 9120 | 9181 | 9242 | 9303 | 9364 | 9425 | 9486 | 9548 | 6 | 12 | 18 | 24 | 31 | 37 | 43 | 49 | 55 |
| 31 | 9610 | 9672 | 9734 | 9797 | 9860 | 9923 | 9986 | | | | 6 | 13 | 19 | 25 | 32 | 38 | 44 | 50 | 57 |
| 31 | | | | | | | | 1005 | 1011 | 1018 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 32 | 1024 | 1030 | 1037 | 1043 | 1050 | 1056 | 1063 | 1069 | 1076 | 1082 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 33 | 1089 | 1096 | 1102 | 1109 | 1116 | 1122 | 1129 | 1136 | 1142 | 1149 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 34 | 1156 | 1163 | 1170 | 1176 | 1183 | 1190 | 1197 | 1204 | 1211 | 1218 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 35 | 1225 | 1232 | 1239 | 1246 | 1253 | 1260 | 1267 | 1274 | 1282 | 1289 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 36 | 1296 | 1303 | 1310 | 1318 | 1325 | 1332 | 1340 | 1347 | 1354 | 1362 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 37 | 1369 | 1376 | 1384 | 1391 | 1399 | 1406 | 1414 | 1421 | 1429 | 1436 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 38 | 1444 | 1452 | 1459 | 1467 | 1475 | 1482 | 1490 | 1498 | 1505 | 1513 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 39 | 1521 | 1529 | 1537 | 1544 | 1552 | 1560 | 1568 | 1576 | 1584 | 1592 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 40 | 1600 | 1608 | 1616 | 1624 | 1632 | 1640 | 1648 | 1656 | 1665 | 1673 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| 41 | 1681 | 1689 | 1697 | 1706 | 1714 | 1722 | 1731 | 1739 | 1747 | 1756 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| 42 | 1764 | 1772 | 1781 | 1789 | 1798 | 1806 | 1815 | 1823 | 1832 | 1840 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| 43 | 1849 | 1858 | 1866 | 1875 | 1884 | 1892 | 1901 | 1910 | 1918 | 1927 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| 44 | 1936 | 1945 | 1954 | 1962 | 1971 | 1980 | 1989 | 1998 | 2007 | 2016 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 45 | 2025 | 2034 | 2043 | 2052 | 2061 | 2070 | 2079 | 2088 | 2098 | 2107 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| 46 | 2116 | 2125 | 2134 | 2144 | 2153 | 2162 | 2172 | 2181 | 2190 | 2200 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 8 |
| 47 | 2209 | 2218 | 2228 | 2237 | 2247 | 2256 | 2266 | 2275 | 2285 | 2294 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 48 | 2304 | 2314 | 2323 | 2333 | 2343 | 2352 | 2362 | 2372 | 2381 | 2391 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 49 | 2401 | 2411 | 2421 | 2430 | 2440 | 2450 | 2460 | 2470 | 2480 | 2490 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 50 | 2500 | 2510 | 2520 | 2530 | 2540 | 2550 | 2560 | 2570 | 2581 | 2591 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 51 | 2601 | 2611 | 2621 | 2632 | 2642 | 2652 | 2663 | 2673 | 2683 | 2694 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 52 | 2704 | 2714 | 2725 | 2735 | 2746 | 2756 | 2767 | 2777 | 2788 | 2798 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 53 | 2809 | 2820 | 2830 | 2841 | 2852 | 2862 | 2873 | 2884 | 2894 | 2905 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| 54 | 2916 | 2927 | 2938 | 2948 | 2959 | 2970 | 2981 | 2992 | 3003 | 3014 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |

# SQUARES

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Differences 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 55 | 3025 | 3036 | 3047 | 3058 | 3069 | 3080 | 3091 | 3102 | 3114 | 3125 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 10 |
| 56 | 3136 | 3147 | 3158 | 3170 | 3181 | 3192 | 3204 | 3215 | 3226 | 3238 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 57 | 3249 | 3260 | 3272 | 3283 | 3295 | 3306 | 3318 | 3329 | 3341 | 3352 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| 58 | 3364 | 3376 | 3387 | 3399 | 3411 | 3422 | 3434 | 3446 | 3457 | 3469 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 9 | 11 |
| 59 | 3481 | 3493 | 3505 | 3516 | 3528 | 3540 | 3552 | 3564 | 3576 | 3588 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 60 | 3600 | 3612 | 3624 | 3636 | 3648 | 3660 | 3672 | 3684 | 3697 | 3709 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| 61 | 3721 | 3733 | 3745 | 3758 | 3770 | 3782 | 3795 | 3807 | 3819 | 3832 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| 62 | 3844 | 3856 | 3869 | 3881 | 3894 | 3906 | 3919 | 3931 | 3944 | 3956 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 63 | 3969 | 3982 | 3994 | 4007 | 4020 | 4032 | 4045 | 4058 | 4070 | 4083 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| 64 | 4096 | 4109 | 4122 | 4134 | 4147 | 4160 | 4173 | 4186 | 4199 | 4212 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| 65 | 4225 | 4238 | 4251 | 4264 | 4277 | 4290 | 4303 | 4316 | 4330 | 4343 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| 66 | 4356 | 4369 | 4382 | 4396 | 4409 | 4422 | 4436 | 4449 | 4462 | 4476 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 67 | 4489 | 4502 | 4516 | 4529 | 4543 | 4556 | 4570 | 4583 | 4597 | 4610 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| 68 | 4624 | 4638 | 4651 | 4665 | 4679 | 4692 | 4706 | 4720 | 4733 | 4747 | 1 | 3 | 4 | 5 | 7 | 8 | 10 | 11 | 12 |
| 69 | 4761 | 4775 | 4789 | 4802 | 4816 | 4830 | 4844 | 4858 | 4872 | 4886 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 70 | 4900 | 4914 | 4928 | 4942 | 4956 | 4970 | 4984 | 4998 | 5013 | 5027 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 13 |
| 71 | 5041 | 5055 | 5069 | 5084 | 5098 | 5112 | 5127 | 5141 | 5155 | 5170 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| 72 | 5184 | 5198 | 5213 | 5227 | 5242 | 5256 | 5271 | 5285 | 5300 | 5314 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 73 | 5329 | 5344 | 5358 | 5373 | 5388 | 5402 | 5417 | 5432 | 5446 | 5461 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 74 | 5476 | 5491 | 5506 | 5520 | 5535 | 5550 | 5565 | 5580 | 5595 | 5610 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| 75 | 5625 | 5640 | 5655 | 5670 | 5685 | 5700 | 5715 | 5730 | 5746 | 5761 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 76 | 5776 | 5791 | 5806 | 5822 | 5837 | 5852 | 5868 | 5883 | 5898 | 5914 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 77 | 5929 | 5944 | 5960 | 5975 | 5991 | 6006 | 6022 | 6037 | 6053 | 6068 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| 78 | 6084 | 6100 | 6115 | 6131 | 6147 | 6162 | 6178 | 6194 | 6209 | 6225 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| 79 | 6241 | 6257 | 6273 | 6288 | 6304 | 6320 | 6336 | 6352 | 6368 | 6384 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 80 | 6400 | 6416 | 6432 | 6448 | 6464 | 6480 | 6496 | 6512 | 6529 | 6545 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| 81 | 6561 | 6577 | 6593 | 6610 | 6626 | 6642 | 6659 | 6675 | 6691 | 6708 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| 82 | 6724 | 6740 | 6757 | 6773 | 6790 | 6806 | 6823 | 6839 | 6856 | 6872 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 83 | 6889 | 6906 | 6922 | 6939 | 6956 | 6972 | 6989 | 7006 | 7022 | 7039 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| 84 | 7056 | 7073 | 7090 | 7106 | 7123 | 7140 | 7157 | 7174 | 7191 | 7208 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 14 | 15 |
| 85 | 7225 | 7242 | 7259 | 7276 | 7293 | 7310 | 7327 | 7344 | 7362 | 7379 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| 86 | 7396 | 7413 | 7430 | 7448 | 7465 | 7482 | 7500 | 7517 | 7534 | 7552 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| 87 | 7569 | 7586 | 7604 | 7621 | 7639 | 7656 | 7674 | 7691 | 7709 | 7726 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 88 | 7744 | 7762 | 7779 | 7797 | 7815 | 7832 | 7850 | 7868 | 7885 | 7903 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| 89 | 7921 | 7939 | 7957 | 7974 | 7992 | 8010 | 8028 | 8046 | 8064 | 8082 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 90 | 8100 | 8118 | 8136 | 8154 | 8172 | 8190 | 8208 | 8226 | 8245 | 8263 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| 91 | 8281 | 8299 | 8317 | 8336 | 8354 | 8372 | 8391 | 8409 | 8427 | 8446 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 15 | 16 |
| 92 | 8464 | 8482 | 8501 | 8519 | 8538 | 8556 | 8575 | 8593 | 8612 | 8630 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 93 | [illegible] | [illegible] | 8686 | 8705 | 8724 | 8742 | 8761 | 8780 | 8798 | 8817 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| 94 | [illegible] | [illegible] | 8874 | 8892 | 8911 | 8930 | 8949 | 8968 | 8987 | 9006 | 2 | 4 | 6 | 8 | 9 | 11 | 13 | 15 | 17 |
| 95 | 9025 | 9044 | 9063 | 9082 | 9101 | 9120 | 9139 | 9158 | 9178 | 9197 | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 |
| 96 | [illegible] | [illegible] | 9254 | 9274 | 9293 | 9312 | 9332 | 9351 | 9370 | 9390 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 97 | [illegible] | [illegible] | 9448 | 9467 | 9487 | 9506 | 9526 | 9545 | 9565 | 9584 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 98 | 9604 | 9624 | 9643 | 9663 | 9683 | 9702 | 9722 | 9742 | 9761 | 9781 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| 99 | 9801 | 9821 | 9841 | 9860 | 9880 | 9900 | 9920 | 9940 | 9960 | 9980 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |

# RECIPROCALS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Subtract Differences 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1·0** | 1·0000 | 9901 | 9804 | 9709 | 9615 | 9524 | 9434 | 9346 | 9259 | 9174 | 9 | 18 | 27 | 36 | 45 | 55 | 64 | 73 | 82 |
| **1·1** | ·9091 | 9009 | 8929 | 8850 | 8772 | 8696 | 8621 | 8547 | 8475 | 8403 | 8 | 15 | 23 | 30 | 38 | 45 | 53 | 61 | 68 |
| **1·2** | ·8333 | 8264 | 8197 | 8130 | 8065 | 8000 | 7937 | 7874 | 7813 | 7752 | 6 | 13 | 19 | 26 | 32 | 38 | 45 | 51 | 58 |
| **1·3** | ·7692 | 7634 | 7576 | 7519 | 7463 | 7407 | 7353 | 7299 | 7246 | 7194 | 5 | 11 | 16 | 22 | 27 | 33 | 38 | 44 | 49 |
| **1·4** | ·7143 | 7092 | 7042 | 6993 | 6944 | 6897 | 6849 | 6803 | 6757 | 6711 | 5 | 10 | 14 | 19 | 24 | 29 | 33 | 38 | 43 |
| **1·5** | ·6667 | 6623 | 6579 | 6536 | 6494 | 6452 | 6410 | 6369 | 6329 | 6289 | 4 | 8 | 13 | 17 | 21 | 25 | 29 | 33 | 38 |
| **1·6** | ·6250 | 6211 | 6173 | 6135 | 6098 | 6061 | 6024 | 5988 | 5952 | 5917 | 4 | 7 | 11 | 15 | 18 | 22 | 26 | 29 | 33 |
| **1·7** | ·5882 | 5848 | 5814 | 5780 | 5747 | 5714 | 5682 | 5650 | 5618 | 5587 | 3 | 6 | 10 | 13 | 16 | 20 | 23 | 26 | 29 |
| **1·8** | ·5556 | 5525 | 5495 | 5464 | 5435 | 5405 | 5376 | 5348 | 5319 | 5291 | 3 | 6 | 9 | 12 | 15 | 18 | 20 | 23 | 26 |
| **1·9** | ·5263 | 5236 | 5208 | 5181 | 5155 | 5128 | 5102 | 5076 | 5051 | 5025 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| **2·0** | ·5000 | 4975 | 4950 | 4926 | 4902 | 4878 | 4854 | 4831 | 4808 | 4785 | 2 | 5 | 7 | 10 | 12 | 14 | 17 | 19 | 21 |
| **2·1** | ·4762 | 4739 | 4717 | 4695 | 4673 | 4651 | 4630 | 4608 | 4587 | 4566 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 17 | 20 |
| **2·2** | ·4545 | 4525 | 4505 | 4484 | 4464 | 4444 | 4425 | 4405 | 4386 | 4367 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **2·3** | ·4348 | 4329 | 4310 | 4292 | 4274 | 4255 | 4237 | 4219 | 4202 | 4184 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| **2·4** | ·4167 | 4149 | 4132 | 4115 | 4098 | 4082 | 4065 | 4049 | 4032 | 4016 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 13 | 15 |
| **2·5** | ·4000 | 3984 | 3968 | 3953 | 3937 | 3922 | 3906 | 3891 | 3876 | 3861 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **2·6** | ·3846 | 3831 | 3817 | 3802 | 3788 | 3774 | 3759 | 3745 | 3731 | 3717 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **2·7** | ·3704 | 3690 | 3676 | 3663 | 3650 | 3636 | 3623 | 3610 | 3597 | 3584 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| **2·8** | ·3571 | 3559 | 3546 | 3534 | 3521 | 3509 | 3497 | 3484 | 3472 | 3460 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| **2·9** | ·3448 | 3436 | 3425 | 3413 | 3401 | 3390 | 3378 | 3367 | 3356 | 3344 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| **3·0** | ·3333 | 3322 | 3311 | 3300 | 3289 | 3279 | 3268 | 3257 | 3247 | 3236 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| **3·1** | ·3226 | 3215 | 3205 | 3195 | 3185 | 3175 | 3165 | 3155 | 3145 | 3135 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **3·2** | ·3125 | 3115 | 3106 | 3096 | 3086 | 3077 | 3067 | 3058 | 3049 | 3040 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **3·3** | ·3030 | 3021 | 3012 | 3003 | 2994 | 2985 | 2976 | 2967 | 2959 | 2950 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **3·4** | ·2941 | 2933 | 2924 | 2915 | 2907 | 2899 | 2890 | 2882 | 2874 | 2865 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **3·5** | ·2857 | 2849 | 2841 | 2833 | 2825 | 2817 | 2809 | 2801 | 2793 | 2786 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **3·6** | ·2778 | 2770 | 2762 | 2755 | 2747 | 2740 | 2732 | 2725 | 2717 | 2710 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **3·7** | ·2703 | 2695 | 2688 | 2681 | 2674 | 2667 | 2660 | 2653 | 2646 | 2639 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| **3·8** | ·2632 | 2625 | 2618 | 2611 | 2604 | 2597 | 2591 | 2584 | 2577 | 2571 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| **3·9** | ·2564 | 2558 | 2551 | 2545 | 2538 | 2532 | 2525 | 2519 | 2513 | 2506 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| **4·0** | ·2500 | 2494 | 2488 | 2481 | 2475 | 2469 | 2463 | 2457 | 2451 | 2445 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| **4·1** | ·2439 | 2433 | 2427 | 2421 | 2415 | 2410 | 2404 | 2398 | 2392 | 2387 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| **4·2** | ·2381 | 2375 | 2370 | 2364 | 2358 | 2353 | 2347 | 2342 | 2336 | 2331 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **4·3** | ·2326 | 2320 | 2315 | 2309 | 2304 | 2299 | 2294 | 2288 | 2283 | 2278 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **4·4** | ·2273 | 2268 | 2262 | 2257 | 2252 | 2247 | 2242 | 2237 | 2232 | 2227 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| **4·5** | ·2222 | 2217 | 2212 | 2208 | 2203 | 2198 | 2193 | 2188 | 2183 | 2179 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **4·6** | ·2174 | 2169 | 2165 | 2160 | 2155 | 2151 | 2146 | 2141 | 2137 | 2132 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **4·7** | ·2128 | 2123 | 2119 | 2114 | 2110 | 2105 | 2101 | 2096 | 2092 | 2088 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| **4·8** | ·2083 | 2079 | 2075 | 2070 | 2066 | 2062 | 2058 | 2053 | 2049 | 2045 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| **4·9** | ·2041 | 2037 | 2033 | 2028 | 2024 | 2020 | 2016 | 2012 | 2008 | 2004 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| **5·0** | ·2000 | 1996 | 1992 | 1988 | 1984 | 1980 | 1976 | 1972 | 1969 | 1965 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| **5·1** | ·1961 | 1957 | 1953 | 1949 | 1946 | 1942 | 1938 | 1934 | 1931 | 1927 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 |
| **5·2** | ·1923 | 1919 | 1916 | 1912 | 1908 | 1905 | 1901 | 1898 | 1894 | 1890 | 0 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 3 |
| **5·3** | ·1887 | 1883 | 1880 | 1876 | 1873 | 1869 | 1866 | 1862 | 1859 | 1855 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| **5·4** | ·1852 | 1848 | 1845 | 1842 | 1838 | 1835 | 1832 | 1828 | 1825 | 1821 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |

# RECIPROCALS

| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Subtract Differences 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 5·5 | ·1818 | 1815 | 1812 | 1808 | 1805 | 1802 | 1799 | 1795 | 1792 | 1789 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 5·6 | ·1786 | 1783 | 1779 | 1776 | 1773 | 1770 | 1767 | 1764 | 1761 | 1757 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| 5·7 | ·1754 | 1751 | 1748 | 1745 | 1742 | 1739 | 1736 | 1733 | 1730 | 1727 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| 5·8 | ·1724 | 1721 | 1718 | 1715 | 1712 | 1709 | 1706 | 1704 | 1701 | 1698 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 5·9 | ·1695 | 1692 | 1689 | 1686 | 1684 | 1681 | 1678 | 1675 | 1672 | 1669 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| 6·0 | ·1667 | 1664 | 1661 | 1658 | 1656 | 1653 | 1650 | 1647 | 1645 | 1642 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 6·1 | 1639 | 1637 | 1634 | 1631 | 1629 | 1626 | 1623 | 1621 | 1618 | 1616 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| 6·2 | ·1613 | 1610 | 1608 | 1605 | 1603 | 1600 | 1597 | 1595 | 1592 | 1590 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·3 | ·1587 | 1585 | 1582 | 1580 | 1577 | 1575 | 1572 | 1570 | 1567 | 1565 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·4 | ·1563 | 1560 | 1558 | 1555 | 1553 | 1550 | 1548 | 1546 | 1543 | 1541 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·5 | 1538 | 1536 | 1534 | 1531 | 1529 | 1527 | 1524 | 1522 | 1520 | 1517 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·6 | ·1515 | 1513 | 1511 | 1508 | 1506 | 1504 | 1502 | 1499 | 1497 | 1495 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·7 | ·1493 | 1490 | 1488 | 1486 | 1484 | 1481 | 1479 | 1477 | 1475 | 1473 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·8 | 1471 | 1468 | 1466 | 1464 | 1462 | 1460 | 1458 | 1456 | 1453 | 1451 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| 6·9 | 1449 | 1447 | 1445 | 1443 | 1441 | 1439 | 1437 | 1435 | 1433 | 1431 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·0 | ·1429 | 1427 | 1425 | 1422 | 1420 | 1418 | 1416 | 1414 | 1412 | 1410 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·1 | 1408 | 1406 | 1404 | 1403 | 1401 | 1399 | 1397 | 1395 | 1393 | 1391 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·2 | 1389 | 1387 | 1385 | 1383 | 1381 | 1379 | 1377 | 1376 | 1374 | 1372 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·3 | 1370 | 1368 | 1366 | 1364 | 1362 | 1361 | 1359 | 1357 | 1355 | 1353 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·4 | 1351 | 1350 | 1348 | 1346 | 1344 | 1342 | 1340 | 1339 | 1337 | 1335 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 |
| 7·5 | 1333 | 1332 | 1330 | 1328 | 1326 | 1325 | 1323 | 1321 | 1319 | 1318 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| 7·6 | ·1316 | 1314 | 1312 | 1311 | 1309 | 1307 | 1305 | 1304 | 1302 | 1300 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 2 |
| 7·7 | ·1299 | 1297 | 1295 | 1294 | 1292 | 1290 | 1289 | 1287 | 1285 | 1284 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7·8 | ·1282 | 1280 | 1279 | 1277 | 1276 | 1274 | 1272 | 1271 | 1269 | 1267 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7·9 | ·1266 | 1264 | 1263 | 1261 | 1259 | 1258 | 1256 | 1255 | 1253 | 1252 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·0 | ·1250 | 1248 | 1247 | 1245 | 1244 | 1242 | 1241 | 1239 | 1238 | 1236 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·1 | ·1235 | 1233 | 1232 | 1230 | 1229 | 1227 | 1225 | 1224 | 1222 | 1221 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·2 | ·1220 | 1218 | 1217 | 1215 | 1214 | 1212 | 1211 | 1209 | 1208 | 1206 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·3 | ·1205 | 1203 | 1202 | 1200 | 1199 | 1198 | 1196 | 1195 | 1193 | 1192 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·4 | ·1190 | 1189 | 1188 | 1186 | 1185 | 1183 | 1182 | 1181 | 1179 | 1178 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·5 | ·1176 | 1175 | 1174 | 1172 | 1171 | 1170 | 1168 | 1167 | 1166 | 1164 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·6 | ·1163 | 1161 | 1160 | 1159 | 1157 | 1156 | 1155 | 1153 | 1152 | 1151 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·7 | ·1149 | 1148 | 1147 | 1145 | 1144 | 1143 | 1142 | 1140 | 1139 | 1138 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·8 | ·1136 | 1135 | 1134 | 1133 | 1131 | 1130 | 1129 | 1127 | 1126 | 1125 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8·9 | ·1124 | 1122 | 1121 | 1120 | 1119 | 1117 | 1116 | 1115 | 1114 | 1112 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9·0 | ·1111 | 1110 | 1109 | 1107 | 1106 | 1105 | 1104 | 1103 | 1101 | 1100 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9·1 | ·1099 | 1098 | 1096 | 1095 | 1094 | 1093 | 1092 | 1091 | 1089 | 1088 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9·2 | ·1087 | 1086 | 1085 | 1083 | 1082 | 1081 | 1080 | 1079 | 1078 | 1076 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·3 | ·1075 | 1074 | 1073 | 1072 | 1071 | 1070 | 1068 | 1067 | 1066 | 1065 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·4 | ·1064 | 1063 | 1062 | 1060 | 1059 | 1058 | 1057 | 1056 | 1055 | 1054 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·5 | ·1053 | 1052 | 1050 | 1049 | 1048 | 1047 | 1046 | 1045 | 1044 | 1043 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·6 | ·1042 | 1041 | 1040 | 1038 | 1037 | 1036 | 1035 | 1034 | 1033 | 1032 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·7 | ·1031 | 1030 | 1029 | 1028 | 1027 | 1026 | 1025 | 1024 | 1022 | 1021 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·8 | ·1020 | 1019 | 1018 | 1017 | 1016 | 1015 | 1014 | 1013 | 1012 | 1011 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 9·9 | ·1010 | 1009 | 1008 | 1007 | 1006 | 1005 | 1004 | 1003 | 1002 | 1001 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |

# DEGREES TO RADIANS

Differences— 1′ 3 | 2′ 6 | 3′ 9 | 4′ 12 | 5′ 15

| ° | Radians | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | ·0000 | 0017 | 0035 | 0052 | 0070 | 0087 | 0105 | 0122 | 0140 | 0157 |
| 1 | ·0175 | 0192 | 0209 | 0227 | 0244 | 0262 | 0279 | 0297 | 0314 | 0332 |
| 2 | ·0349 | 0367 | 0384 | 0401 | 0419 | 0436 | 0454 | 0471 | 0489 | 0506 |
| 3 | ·0524 | 0541 | 0559 | 0576 | 0593 | 0611 | 0628 | 0646 | 0663 | 0681 |
| 4 | ·0698 | 0716 | 0733 | 0750 | 0768 | 0785 | 0803 | 0820 | 0838 | 0855 |
| 5 | ·0873 | 0890 | 0908 | 0925 | 0942 | 0960 | 0977 | 0995 | 1012 | 1030 |
| 6 | ·1047 | 1065 | 1082 | 1100 | 1117 | 1134 | 1152 | 1169 | 1187 | 1204 |
| 7 | ·1222 | 1239 | 1257 | 1274 | 1292 | 1309 | 1326 | 1344 | 1361 | 1379 |
| 8 | ·1396 | 1414 | 1431 | 1449 | 1466 | 1484 | 1501 | 1518 | 1536 | 1553 |
| 9 | ·1571 | 1588 | 1606 | 1623 | 1641 | 1658 | 1676 | 1693 | 1710 | 1728 |
| 10 | ·1745 | 1763 | 1780 | 1798 | 1815 | 1833 | 1850 | 1868 | 1885 | 1902 |
| 11 | ·1920 | 1937 | 1955 | 1972 | 1990 | 2007 | 2025 | 2042 | 2059 | 2077 |
| 12 | ·2094 | 2112 | 2129 | 2147 | 2164 | 2182 | 2199 | 2217 | 2234 | 2251 |
| 13 | ·2269 | 2286 | 2304 | 2321 | 2339 | 2356 | 2374 | 2391 | 2409 | 2426 |
| 14 | ·2443 | 2461 | 2478 | 2496 | 2513 | 2531 | 2548 | 2566 | 2583 | 2601 |
| 15 | ·2618 | 2635 | 2653 | 2670 | 2688 | 2705 | 2723 | 2740 | 2758 | 2775 |
| 16 | ·2793 | 2810 | 2827 | 2845 | 2862 | 2880 | 2897 | 2915 | 2932 | 2950 |
| 17 | ·2967 | 2985 | 3002 | 3019 | 3037 | 3054 | 3072 | 3089 | 3107 | 3124 |
| 18 | ·3142 | 3159 | 3176 | 3194 | 3211 | 3229 | 3246 | 3264 | 3281 | 3299 |
| 19 | ·3316 | 3334 | 3351 | 3368 | 3386 | 3403 | 3421 | 3438 | 3456 | 3473 |
| 20 | ·3491 | 3508 | 3526 | 3543 | 3560 | 3578 | 3595 | 3613 | 3630 | 3648 |
| 21 | ·3665 | 3683 | 3700 | 3718 | 3735 | 3752 | 3770 | 3787 | 3805 | 3822 |
| 22 | ·3840 | 3857 | 3875 | 3892 | 3910 | 3927 | 3944 | 3962 | 3979 | 3997 |
| 23 | ·4014 | 4032 | 4049 | 4067 | 4084 | 4102 | 4119 | 4136 | 4154 | 4171 |
| 24 | ·4189 | 4206 | 4224 | 4241 | 4259 | 4276 | 4294 | 4311 | 4328 | 4346 |
| 25 | ·4363 | 4381 | 4398 | 4416 | 4433 | 4451 | 4468 | 4485 | 4503 | 4520 |
| 26 | ·4538 | 4555 | 4573 | 4590 | 4608 | 4625 | 4643 | 4660 | 4677 | 4695 |
| 27 | ·4712 | 4730 | 4747 | 4765 | 4782 | 4800 | 4817 | 4835 | 4852 | 4869 |
| 28 | ·4887 | 4904 | 4921 | 4939 | 4957 | 4974 | 4992 | 5009 | 5027 | 5044 |
| 29 | ·5061 | 5079 | 5096 | 5114 | 5131 | 5149 | 5166 | 5184 | 5201 | 5219 |
| 30 | ·5236 | 5253 | 5271 | 5288 | 5306 | 5323 | 5341 | 5358 | 5376 | 5393 |
| 31 | ·5411 | 5428 | 5445 | 5463 | 5480 | 5498 | 5515 | 5533 | 5550 | 5568 |
| 32 | ·5585 | 5603 | 5620 | 5637 | 5655 | 5672 | 5690 | 5707 | 5725 | 5742 |
| 33 | ·5760 | 5777 | 5794 | 5812 | 5829 | 5847 | 5864 | 5882 | 5899 | 5917 |
| 24 | ·5934 | 5952 | 5969 | 5986 | 6004 | 6021 | 6039 | 6056 | 6074 | 6091 |
| 35 | ·6109 | 6126 | 6144 | 6161 | 6178 | 6196 | 6213 | 6231 | 6248 | 6266 |
| 36 | ·6283 | 6301 | 6318 | 6336 | 6353 | 6370 | 6388 | 6405 | 6423 | 6440 |
| 37 | ·6458 | 6475 | 6493 | 6510 | 6528 | 6545 | 6562 | 6580 | 6597 | 6615 |
| 38 | ·6632 | 6650 | 6667 | 6685 | 6702 | 6720 | 6737 | 6754 | 6772 | 6789 |
| 39 | ·6807 | 6824 | 6842 | 6859 | 6877 | 6894 | 6912 | 6929 | 6946 | 6964 |
| 40 | ·6981 | 6999 | 7016 | 7034 | 7051 | 7069 | 7086 | 7103 | 7121 | 7138 |
| 41 | ·7156 | 7173 | 7191 | 7208 | 7226 | 7243 | 7261 | 7278 | 7295 | 7313 |
| 42 | ·7330 | 7348 | 7365 | 7383 | 7400 | 7418 | 7435 | 7453 | 7470 | 7487 |
| 43 | ·7505 | 7522 | 7540 | 7557 | 7575 | 7592 | 7610 | 7627 | 7645 | 7662 |
| 44 | ·7679 | 7697 | 7714 | 7732 | 7749 | 7767 | 7784 | 7802 | 7819 | 7837 |

# DEGREES TO RADIANS

| Differences— | 1′ | 2′ | 3′ | 4′ | 5′ |
|---|---|---|---|---|---|
| | 3 | 6 | 9 | 12 | 15 |

| ° | Radians | 6′ | 12′ | 18′ | 24′ | 30′ | 36′ | 42′ | 48′ | 54′ |
|---|---|---|---|---|---|---|---|---|---|---|
| 45 | ·7854 | 7871 | 7889 | 7906 | 7924 | 7941 | 7959 | 7976 | 7994 | 8011 |
| 46 | ·8029 | 8046 | 8063 | 8081 | 8098 | 8116 | 8133 | 8151 | 8168 | 8186 |
| 47 | ·8203 | 8221 | 8238 | 8255 | 8273 | 8290 | 8308 | 8325 | 8343 | 8360 |
| 48 | ·8378 | 8395 | 8412 | 8430 | 8447 | 8465 | 8482 | 8500 | 8517 | 8535 |
| 49 | ·8552 | 8570 | 8587 | 8604 | 8622 | 8639 | 8657 | 8674 | 8692 | 8709 |
| 50 | ·8727 | 8744 | 8762 | 8779 | 8796 | 8814 | 8831 | 8849 | 8866 | 8884 |
| 51 | ·8901 | 8919 | 8936 | 8954 | 8971 | 8988 | 9006 | 9023 | 9041 | 9058 |
| 52 | ·9076 | 9093 | 9111 | 9128 | 9146 | 9163 | 9180 | 9198 | 9215 | 9233 |
| 53 | ·9250 | 9268 | 9285 | 9303 | 9320 | 9338 | 9355 | 9372 | 9390 | 9407 |
| 54 | ·9425 | 9442 | 9460 | 9477 | 9495 | 9512 | 9529 | 9547 | 9564 | 9582 |
| 55 | ·9599 | 9617 | 9634 | 9652 | 9669 | 9687 | 9704 | 9721 | 9739 | 9756 |
| 56 | ·9774 | 9791 | 9809 | 9826 | 9844 | 9861 | 9879 | 9896 | 9913 | 9931 |
| 57 | ·9948 | 9966 | 9983 | $\bar{0}001$ | $\bar{0}018$ | $\bar{0}036$ | $\bar{0}053$ | $\bar{0}071$ | $\bar{0}088$ | $\bar{0}105$ |
| 58 | 1·0123 | 0140 | 0158 | 0175 | 0193 | 0210 | 0228 | 0245 | 0263 | 0280 |
| 59 | 1·0297 | 0315 | 0332 | 0350 | 0367 | 0385 | 0402 | 0420 | 0437 | 0455 |
| 60 | 1·0472 | 0489 | 0507 | 0524 | 0542 | 0559 | 0577 | 0594 | 0612 | 0629 |
| 61 | 1·0647 | 0664 | 0681 | 0699 | 0716 | 0734 | 0751 | 0769 | 0786 | 0804 |
| 62 | 1·0821 | 0838 | 0856 | 0873 | 0891 | 0908 | 0926 | 0943 | 0961 | 0978 |
| 63 | 1·0996 | 1013 | 1030 | 1048 | 1065 | 1083 | 1100 | 1118 | 1135 | 1153 |
| 64 | 1·1170 | 1188 | 1205 | 1222 | 1240 | 1257 | 1275 | 1292 | 1310 | 1327 |
| 65 | 1·1345 | 1362 | 1380 | 1397 | 1414 | 1432 | 1449 | 1467 | 1484 | 1502 |
| 66 | 1·1519 | 1537 | 1554 | 1572 | 1589 | 1606 | 1624 | 1641 | 1659 | 1676 |
| 67 | 1·1694 | 1711 | 1729 | 1746 | 1764 | 1781 | 1798 | 1816 | 1833 | 1851 |
| 68 | 1·1868 | 1886 | 1903 | 1921 | 1938 | 1956 | 1973 | 1990 | 2008 | 2025 |
| 69 | 1·2043 | 2060 | 2078 | 2095 | 2113 | 2130 | 2147 | 2165 | 2182 | 2200 |
| 70 | 1·2217 | 2235 | 2252 | 2270 | 2287 | 2305 | 2322 | 2339 | 2357 | 2374 |
| 71 | 1·2392 | 2409 | 2427 | 2444 | 2462 | 2479 | 2497 | 2514 | 2531 | 2549 |
| 72 | 1·2566 | 2584 | 2601 | 2619 | 2636 | 2654 | 2671 | 2689 | 2706 | 2723 |
| 73 | 1·2741 | 2758 | 2776 | 2793 | 2811 | 2828 | 2846 | 2863 | 2881 | 2898 |
| 74 | 1·2915 | 2933 | 2950 | 2968 | 2985 | 3003 | 3020 | 3038 | 3055 | 3073 |
| 75 | 1·3090 | 3107 | 3125 | 3142 | 3160 | 3177 | 3195 | 3212 | 3230 | 3247 |
| 76 | 1·3265 | 3282 | 3299 | 3317 | 3334 | 3352 | 3369 | 3387 | 3404 | 3422 |
| 77 | 1·3439 | 3456 | 3474 | 3491 | 3509 | 3526 | 3544 | 3561 | 3579 | 3596 |
| 78 | 1·3614 | 3631 | 3648 | 3666 | 3683 | 3701 | 3718 | 3736 | 3753 | 3771 |
| 79 | 1·3788 | 3806 | 3823 | 3840 | 3858 | 3875 | 3893 | 3910 | 3928 | 3945 |
| 80 | 1·3963 | 3980 | 3998 | 4015 | 4032 | 4050 | 4067 | 4085 | 4102 | 4120 |
| 81 | 1·4137 | 4155 | 4172 | 4190 | 4207 | 4224 | 4242 | 4259 | 4277 | 4294 |
| 82 | 1·4312 | 4329 | 4347 | 4364 | 4382 | 4399 | 4416 | 4434 | 4451 | 4469 |
| 83 | 1·4486 | 4504 | 4521 | 4539 | 4556 | 4573 | 4591 | 4608 | 4626 | 4643 |
| 84 | 1·4661 | 4678 | 4696 | 4713 | 4731 | 4748 | 4765 | 4783 | 4800 | 4818 |
| 85 | 1·4835 | 4853 | 4870 | 4888 | 4905 | 4923 | 4940 | 4957 | 4975 | 4992 |
| 86 | 1·5010 | 5027 | 5045 | 5062 | 5080 | 5097 | 5115 | 5132 | 5149 | 5167 |
| 87 | 1·5184 | 5202 | 5219 | 5237 | 5254 | 5272 | 5289 | 5307 | 5324 | 5341 |
| 88 | 1·5359 | 5376 | 5394 | 5411 | 5429 | 5446 | 5464 | 5481 | 5499 | 5516 |
| 89 | 1·5533 | 5551 | 5568 | 5586 | 5603 | 5621 | 5638 | 5656 | 5673 | 5691 |